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Statistical Convergence Estimates for *(p, q)*-Baskakov-Durrmeyer Type Operators

Prerna Sharma

Department of Basic Science
Sardar Vallabh Bhai Patel University of Agriculture and Technology,
Meerut (UP), *India*
E-mail:*mprerna_anand@yahoo.com*

Abstract: *This paper concerns with the study of(p, q)-analogue of genuine Baskakov- Durrmeyer type operators. We establish the direct approximation theorem, a weighted approximation theorem followed by the estimations of the rate of convergence of these operators for functions of polynomial growth on the interval* $[0, \infty)$.

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1. Introduction

In the theory of approximation, the quantum calculus has been studied for a long time. Quantum calculus was started by the well known mathematician Lupus [11], when he first proposed the q-variant of the Bernstein polynomials. T. Kim gave his contribution on q-type of polynomial in [9], [10]. In the same notions similar type of results on q-analogue of linear positive operators were obtained by [19], [20], [21] etc. Present paper deals with *(p, q)*-calculus (post-quantum calculus), which is an advanced extension of quantum calculus. Mursaleen et al. [12], introduced the Bernstein polynomials using (p, q)-calculus, which was further improved in [13].
(p, q)-calculus was introduced by the classical work of Sahai and Yadav [25]. Recently, a lot of work on *(p, q)*-version of linear positive operators has been published in Acer et al. [2] [1], Aral and Gupta [5], Gupta [8], Mursaleen et al. [14] [15]. We also consider some more results on approximation of functions by positive linear operators using *(p, q)*-calculus given in ([3],[16],[17]).
P. Maheshwari and M. Abid [18] published a paper on approximation of *(p, q)* Szasz-Beta-Stancu operators. To recall some definition and notations of *(p, q)*-calculus, we refer to authors [22], [23] and [24].
The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \qquad n = 0, 1, 2 \dots and\ [0]_{p,q} = 0.$$

The *(p, q)*-factorial $[n]p, q!$ is defined as

$$[n]p, q! = \prod_{k=1}^{n} [k]_{p,q}, \qquad n \geq 1,\ \ [0]p, q! = 1.$$

The *(p, q)*-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]p, q!}{[n-k]p, q!\,[k]p, q!},\ \ 0 \leq k \leq n.$$

(p, q)-derivative is given as

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x},\ x \neq 0.$$

The *(p, q)*-power basis is defined below

$$(x \oplus y)_{p,q}^{n} = (x+y)(px+qy)(p^2x+q^2y)\dots(p^{n-1}x+q^{n-1}y)$$

$(x \ominus y)_{p,q}^{n} = (x-y)(px-qy)(p^2x-q^2y)\dots(p^{n-1}x-q^{n-1}y).$

We propose *(p, q)*-Beta function as,

$$B_{p,q}(m,n) = p^{n/2}q^{m/2}\int_0^{\infty/A}\frac{x^{m-1}}{(1+\oplus x)_{p,q}^{n+m}}d_{p,q}t, \quad m,n \in N, \tag{1}$$

(p,q)-Gamma function is defined as

$$\Gamma_{p,q}(n+1) = \frac{(p\ominus q)_{p,q}^{n}}{(p-q)^n} = [n]_{p,q}!, \quad 0<q<p. \tag{2}$$

Proposition 1. [23] The *(p, q)*-integration by parts is given by

$$\int_a^b g(px)\,D_{p,q}h(x)d_{p,q}x = g(b)h(b) - g(a)h(a) - \int_a^b h(qx)\,D_{p,q}g(x)d_{p,q}x.$$

The *(p, q)*-Beta function of second kind [5] is given by

$B_{p,q}(m,n) = p^{n/2}q^{m/2}\int_0^{\infty}\frac{x^{m-1}}{(1+\oplus px)_{p,q}^{n+m}}d_{p,q}x,$ where $m,n \in N$.

The relation between *(p, q)*-Beta and *(p, q)*-Gamma functions is given as

$$B_{p,q}(m,n) = q^{\frac{2-m(m-1)}{2}}p^{\frac{-m(m+1)}{2}}\frac{\Gamma_{p,q}m\Gamma_{p,q}n}{\Gamma_{p,q}(m+n)}.$$

To approximate Lebesgue integrable function on the interval $[0, \infty)$, Agrawal and Thamar [4] introduced the following operators, which is an extension of Srivastava-Gupta operators [25],

$$G_n(f,x) = (n-1)\sum_{k=1}^{\infty}s_{n,k}(x)\int_0^{\infty}s_{n,k-1}(t)f(t)dt + p_{n,0}f(0), \tag{3}$$

where

$$s_{n,k}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{(1+x)^{n+k}}.$$

We introduce the *(p, q)*-analogue of genuine Baskakov-Durrmeyer operators for$x \in [0,\infty)$ and $0 < q < p \leq 1$,the operators are defined as

$$G_n^{p,q}(f,x) = [n-1]_{p,q}\sum_{k=1}^{\infty}s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)} \times \int_0^{\infty/A}s_{n,k-1}^{p,q}(t)f(t)d_{p,q}t + p_{n,0}^{p,q}(x)f(0) \tag{4}$$

where $s_{n,k}^{p,q}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q}\frac{x^k}{(1\oplus x)_{p,q}^{n+k}}$.

It can be noted here, if we put $p = q = 1$, we get well known Baskakov Durrmeyer operators.

2. Auxiliary Results

In this section, we establish some basic results to prove our main theorems.

Lemma 1. For $x \in [0,\infty)$ and $0<q<p\leq 1$, we have

$$G_n^{p,q}(1,x) = 1$$

$$G_n^{p,q}(t,x) = \frac{[n]_{p,q}x}{pq[n-2]_{p,q}}$$

$$G_n^{p,q}(t^2,x) = \frac{[n]_{p,q}x^2}{p^2q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{[n]_{p,q}[2]_{p,q}x}{p^{-n+4}q^3[n-2]_{p,q}[n-3]_{p,q}}.$$

Proof. By the definition *(p, q)*-Beta function given in (1), we get the following estimates

(i) For *f(t)* = 1, we have

$$G_n^{p,q}(1,x) = [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\int_0^{\infty/A} s_{n,k-1}^{p,q}(t)d_{p,q}t + s_{n,0}^{p,q}(x)$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\int_0^{\infty/A}\frac{t^{k-1}}{(1\oplus x)_{p,q}^{n+k-1}}d_{p,q}t + s_{n,0}^{p,q}(x)$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\frac{B_{p,q}(k,n-1)}{p^{(n-1)/2}q^{k/2}} + s_{n,0}^{p,q}(x)$$

$$= \sum_{k=0}^{\infty} p^{\frac{n}{2}}q^{\frac{k}{2}}s_{n,k}^{p,q}(px) = 1.$$

(ii) For $f(t) = t$, we have

$$G_n^{p,q}(t,x) = [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\int_0^{\infty/A} s_{n,k-1}^{p,q}(t)t\,d_{p,q}t$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\int_0^{\infty/A}\frac{t^{k}}{(1\oplus t)_{p,q}^{n+k-1}}d_{p,q}t$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\frac{B_{p,q}(k+1,n-2)}{p^{(n-2)/2}q^{(k+1)/2}}$$

$$= \frac{1}{[n-2]_{p,q}}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)\frac{p^{(n-1)^2+k}q^{k(k-1)}}{p^{\frac{n-2}{2}}q^{\frac{k+1}{2}}}[k]_{p,q}$$

$$= \frac{[n]_{p,q}x}{pq[n-2]_{p,q}}\sum_{k=0}^{\infty} p^{(n+1)/2}q^{k/2}s_{n+1,k}^{p,q}(px) = \frac{[n]_{p,q}x}{pq[n-2]_{p,q}}.$$

(iii) When $f(t) = t^2$, we get

$$G_n^{p,q}(t^2,x) = [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\int_0^{\infty/A} s_{n,k-1}^{p,q}(t)t^2d_{p,q}t$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\int_0^{\infty/A}\frac{t^{k+1}}{(1\oplus t)_{p,q}^{n+k-1}}d_{p,q}t$$

$$= [n-1]_{p,q}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)p^{(n-1)^2+k}q^{k(k-1)}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_{p,q}\frac{B_{p,q}(k+2,n-2)}{p^{(n-3)/2}q^{(k+2)/2}}$$

$$= \frac{1}{[n-2]_{p,q}[n-3]_{p,q}}\sum_{k=1}^{\infty} s_{n,k}^{p,q}(x)\frac{p^{(n-1)^2+k}q^{k(k-1)}}{p^{\frac{n-3}{2}}q^{\frac{k+2}{2}}}[k]_{p,q}[k+1]_{p,q}$$

$$= \frac{[n]_{p,q}x^2}{p^2q^4[n-2]_{p,q}[n-3]_{p,q}}\sum_{k=0}^{\infty} p^{(n+2)/2}q^{k/2}s_{n+2,k}^{p,q}(px)$$

$$= \frac{[2]_{p,q}[n]_{p,q}x}{p^{-n+4}q^3[n-2]_{p,q}[n-3]_{p,q}}\sum_{k=0}^{\infty} p^{(n+1)/2}q^{k/2}s_{n+1,k}^{p,q}(px)$$

$$= \frac{[n]_{p,q}x^2}{p^2q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{[n]_{p,q}[2]_{p,q}x}{p^{-n+4}q^3[n-2]_{p,q}[n-3]_{p,q}}.$$

Lemma 2. For $0 < q < p \leq 1$, we have the following explicit formulae for the central moments

(i) $G_n^{p,q}((t-x),x) = \frac{(1-pq)[n]_{p,q}x+[2]_{p,q}xpq}{pq[n-2]_{p,q}}$

(ii) $G_n^{p,q}((t-x)^2,x) = \left(\frac{[n]_{p,q}[n+1]_{p,q}}{p^2q^4[n-2]_{p,q}[n-3]_{p,q}} - \frac{[n]_{p,q}[2]_{p,q}}{pq[n-2]_{p,q}} + 1\right)x^2 + \frac{[n]_{p,q}[2]_{p,q}x}{p^{-n+4}q^3[n-2]_{p,q}[n-3]_{p,q}}$.

Remark 1. For $0 < q < 1$and $q < p \leq 1$, we may have that $\lim_{n\to\infty}[n]_{p,q} = \frac{1}{(q-p)}$.

To find the convergence of (p,q)-Baskakov-Durrmeyer operators, we consider $p = p_n$ and$q = q_n$are such that $0 < q_n < p_n \leq 1$ andfor sufficiently large $n, p_n \to 1,\ q_n \to 1,\ p_n^n \to a, q_n^n \to b$and $[n]_{p_n,q_n} \to \infty$.

3. Main Results

Definition 1: Let $C_{x^2}[0,\infty)$, be the class of all function *f*, which are defined on the positive real axis and satisfy$|f(x)| \leq C(1+x^2)$, where C is a positive constant depending on *f*. By $C_{x^2}[0,\infty)$, we mean, the subspaceof all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The class $C_{x^2}^*[0,\infty)$is endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

Theorem 1. Let $p = p_n$ and $q = q_n$ satisfy $0 < q_n < p_n \leq 1$ and for sufficiently large $n,\ p_n \to 1,\ q_n \to 1$, then for each $f \in C_{x^2}^*[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|G_n^{p_n,q_n}(f) - f\right\|_{x^2} = 0.$$

Proof. By ([7], Theorem 4.1.4), it is sufficient to show that

$$\lim_{n\to\infty}\left\|G_n^{p_n,q_n}(t^k,x) - x^k\right\|_{x^2} = 0,\ k = 0,1,2. \quad \mathbf{(3.1)}$$

Since $G_n^{p_n,q_n}(1,x) = 1$, the first condition of (3.1) is satisfied for *k = 0.*

In view of Lemma 1, we have

$$\left\|G_n^{p_n,q_n}(t,x) - x\right\|_{x^2} = \sup_{x\in[0,\infty)}\frac{\left|G_n^{p_n,q_n}(t,x) - x\right)|}{1+x^2}$$

$$\leq \left(\frac{[n]_{p_n,q_n}}{p_nq_n[n-2]_{p_n,q_n}} - 1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} \leq \left(\frac{[n]_{p_n,q_n}}{p_nq_n[n-2]_{p_n,q_n}} - 1\right).$$

Taking limit on both the sides as $n \to \infty$,

$$\lim_{n\to\infty}\left\|G_n^{p_n,q_n}(t,x) - x\right\|_{x^2} = 0.$$

The condition of (3.1) is satisfied for *k*=1.

Again using Lemma 1, we obtain

$$\left\|G_n^{p_n,q_n}(t^2,x) - x^2\right\|_{x^2} = \sup_{x\in[0,\infty)}\frac{\left|G_n^{p_n,q_n}(t^2,x) - x^2\right)|}{1+x^2}$$

$$\leq \left(\frac{[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{{p_n}^2{q_n}^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} - 1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}$$

$$+\left(\frac{[n]_{p_n,q_n}[2]_{p_n,q_n}}{{p_n}^{-n+4}{q_n}^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$\leq \left(\frac{[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{{p_n}^2{q_n}^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} - 1\right) + \left(\frac{[n]_{p_n,q_n}[2]_{p_n,q_n}}{{p_n}^{-n+4}{q_n}^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right),$$

which implies that

$$\lim_{n\to\infty}\left\|G_n^{p_n,q_n}(t^2,x) - x^2\right\|_{x^2} = 0.$$

Thus the proof is completed.

Definition 2: Let $C_B[0, \infty)$ be the space of all real valued uniformly continuous and bounded function f on the interval$[0, \infty)$. For $f \in C_B[0, \infty)$ the Peetre's K-functional is defined as

$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\|; g \in C_B^2[0,\infty)\}$.

where $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0, \infty); g', g'' \in C_B[0, \infty)\}$. By Devore and Lorentz [6], there exists anabsolute constant $P > 0$ such that

$$K_2(f,\delta) \le P\omega_2(f,\sqrt{\delta}),$$

where the second order modulus of continuity is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{0\le x\le\infty} |f(x+2h) - 2f(x+h) + f(x)|,$$

and the usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{0\le x\le\infty} |f(x+h) - f(x)|.$$

Theorem 2. Let $f \in C_B[0, \infty)$ and $x \ge 0$, then there exists a constant $P > 0$, such that

$$\left\|G_n^{p_n,q_n}(f,x) - f(x)\right\| \le P\omega_2\left(f, \sqrt{\delta_n^{p_n,q_n}(x)}\right),$$

where

$$\delta_n^{p_n,q_n}(x) = \left(\frac{[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{{p_n}^2{q_n}^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} - \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}}{p_nq_n[n-2]_{p_n,q_n}} + 1\right)x^2 + \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}x}{{p_n}^{-n+4}{q_n}^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}.$$

Proof. Let $g \in C_B^2[0,\infty)$, then by Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-k)g''(k)dk, \quad t \in [0,\infty),$$

which implies that

$$G_n^{p_n,q_n}(g,x) - g(x) = G_n^{p_n,q_n}\left(\int_x^t (t-k)g''(k)dk; x\right).$$

Hence

$$\left|G_n^{p_n,q_n}(g,x) - g(x)\right| \le G_n^{p_n,q_n}(t-x)^2\|g''\|$$
$$= \left[\left(\frac{[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{{p_n}^2{q_n}^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} - \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}}{p_nq_n[n-2]_{p_n,q_n}} + 1\right)x^2 + \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}x}{{p_n}^{-n+4}{q_n}^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right]\|g''\|.$$

Also by Lemma 1, we have

$$\left|G_n^{p_n,q_n}(f,x)\right| \le \|f\|.$$

Therefore, we have

$$\left|G_n^{p_n,q_n}(f,x) - f(x)\right| \le \left|G_n^{p_n,q_n}(f-g,x) - (f-g)(x) + \left|G_n^{p_n,q_n}(g,x) - g(x)\right|\right|$$
$$\le 2\|f-g\| + \left[\left(\frac{[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{{p_n}^2{q_n}^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} - \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}}{p_nq_n[n-2]_{p_n,q_n}} + 1\right)x^2 + \frac{[n]_{p_n,q_n}[2]_{p_n,q_n}x}{{p_n}^{-n+4}{q_n}^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right]\|g''\|.$$

Taking the infimum on the right hand side over all $g \in C_B^2[0,\infty)$and applying the Peetre's K-functional, we get the required result.

$$\left\|G_n^{p_n,q_n}(f,x) - f(x)\right\| \le P\omega_2\left(f, \sqrt{\delta_n^{p_n,q_n}(x)}\right).$$

References

[1] Acar, A., Aral, A. and Mohiuddine, S. A. **(2016)**. On Kantorovich modification of *(p, q)*-Baskakov Operators. J. *Inequal. Appl*., (98): DOI: 10.1186/s13660-016-1045-9

[2] Acar, T., Mohiuddine,S. A. andMursaleen, M. **(2018)**. Approximation by *(p, q)*-Baskakov-Durrmeyer-Stancu operators. *Complex Anal. Oper. Theory*, **12**(6): 1453-1468.

[3] Ansari, K. J. and Karaisa, A. **(2017)**. On the approximation by Chlodowsky type generalization of *(p,q)*-Bernstien operators. *Int. J. Nonlinear Anal. Appl.* , **8**(2): 181-200.

[4] Agrawal,P. N. and Thamar, K. J. **(1998)**. Approximation of unbounded functions by a new sequence of linear positive operatorss. *Math.Anal. Appl.* , **225**: 660-672.

[5] Aral, A. and Gupta, V. (2016).*(p, q)*-type Beta functions of second kind.*Adv. Oper. Theory*, **1**(1): 134-146.

[6] Devore, R. A. and Lorentz, G. G. **(1993)**.*Constructive Approximation*, Springer, Berlin.

[7] Gad˜zhiev, A. D.**(1976)**. Theorem of the type of P. P Korovkin type theorems, presented at the international conference on the theory of approximation of functions Mat. Zametki, **20**(5): 781-786.

[8] Gupta, V. **(2018)**. *(p, q)*-Szasz-Mirakyan-Baskakov operators. *Complex Anal. Oper. Theory*, **12**(1): 17-25.

[9] Kim,T. **(2007)**. q-Euler numbers and polynomials associated with p-adic q-integrals.*J. of Nonlinear Math. Phys*.,**14**(1): 15-27.

[10] Kim, T. **(2011)**. A note on q-Bernstein polynomials. *Russ. J. Math. Phys.* , **18**(73).

[11] Lupas, A. **(1987)**. A q-analogue of the Bernstein operators. In: Seminar on Numeri. and Stat. Calculus , **9**: 85-92.

[12] Mursaleen, M., Ansari K. J. and Khan A. **(2015)**. On *(p, q)*-analogue of Bernstein operators. *Appl. Math. Comput*.,**266**: 874-882.

[13] Mursaleen, M., Ansari K. J. and Khan A. **(2016)**. On *(p, q)*-analogue of Bernstein operators. *Erratum: Appl. Math. Comput*., **278**: 70-71).

[14] Mursaleen, M., Ansari, K. J. and Khan, A. **(2015)**. Some approximation results by *(p, q)*-analogue of Bernstein-Stancu operators. *Appl.Math. Comput.* , **264**: 392-402.

[15] Mursaleen, M., Nasiruzzaman, M.,. Khan, A. and Ansari, K. J. **(2016)**. Some approximation results on Bleimann-Butzer-Hahn operators defined by *(p, q)*-integers. *Filomat*, **30**(3): 639-648.

[16] Mursaleen, M., Nasiruzzaman, M., Ansari, K. J. and Alotaibi, A. **(2017)**. Generalized *(p, q)*-Bleimann-Butzer-Hahn operators and some approximation results. *J. Ineq. Appl.*, **2017**(310): 2017, DOI 10.1186/s13660-017-1582-x.

[17] Mursleen , M., Alotaibi , A. and Ansari, K. J. **(2016)**. On a Kantrovich variant of *(p, q)*-Szasz-Mirakjan Operators. *J. Func. Space*, Article ID 1035253, 9 pages, http:dx.doi.org/10.1155/2016/1035253.

[18] Maheshwari, P.and Abid, M. **(2020)**. Approximation by *(p, q)*-Szasz-Beta-Stancu operators operators. *Arabian J. of Mathematics*, **9**: 191-200.DOI 10.1007/s40065-019-0240-1.

[19] Maheshwari, P. **(2018)**. Approximation properties of certain q-genuine Szasz operators. *Complex Anal. Oper. Theory*, **12**(1): 27-36.

[20] Maheshwari, P. and Sharma, D. **(2012)**. Approximation by q-Baskakov-Beta-Stancu operators. *Rend. Circ. Mat. Palermo.*, **61**:297-305.

[21] Prakash, O., Sharma, D. and Maheshwari, P. **(2015)**. Certain generalized q-operators. *Demonstratio Mathematica*, **3**(48): 404-412.

[22] Sadjang, P. N.**(2015)**. On the *(p,* q)-Gamma and the *(p, q)*-Beta function, arXiv 1506, 0739v1 [Math NT]

[23] Sadjang, P. N. **(2013)**. On the fundamental theorem of *(p, q)*-calculus and some *(p, q)-Taylor formulas. arXiv:* 1309.3934*v*1 [*Math.QA*]

[24] Sahai, V. and Yadav, S. **(2007)**. Representations of two parameter quantum algebras and *(p, q)*-special functions. *J. Math. Anal. Appl.* , **335**(1): 268-279.

[25] Srivastava, H. M. and Gupta, V. **(2003)**. A certain family of summation-integral type operators. *Math. Comput. Model*, **37:**1307-1315.