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On Generalized h −Recurrent Finsler Connection

Khageshwar Mandal

Department of Mathematics,

Padma Kanya Campus, *Tribhuvan University*, *Kathmandu*, *Nepal*

Email: *khageshwarmandal@gmail.com*

Abstract:
The purpose of the present paper is to generalize the concept of recurrent Finsler connection by taking h-connection by applying h-covariant derivative of $\emptyset_{ij}^{(p)}$ as recurrent. Such a connection will be called a generalized h-recurrent Finsler connection. The relation between curvature tensors of Cartan's connection $C\Gamma$ and generalized h-recurrent Finsler connection has been established.

Keywords: *Finsler connection*, *Cartan's connection*, *Curvature tensor*, *Torsion tensor*, *Deflection tensor*.

1. Introduction

Cartan (Cartan,1994) published his monograph 'les espaces de Finsler' and fixed his method to determine a notion of connection in the Geometry of Finsler space. Matsumoto (Matsumoto, 2008) determined uniquely the Cartan's connection CT by following conditions: (1) the connection is metrical; (2) the deflection tensor field vanishes; (3) the torsion tensor field T vanishes; (4) the torsion tensor field S vanishes.

Hozo (Hojo, 2016, Hashiguchi, 1998) introduced the connections, which depend on a real parameter P and make v-covariant derivative $\emptyset_{ij}^{(p)}\|_k$ of $\emptyset_{ij}^{(p)}(=\dot{\partial}_i\dot{\partial}_j L^P)$zero just like as $g_{ij}|_k = 0$ in case of $C\Gamma$. The cartan's connection is really the case when P takes its value two and so the connection determined by Hozo is a generalization of $C\Gamma$.

Recently Prasad and Srivastava (2012) have introduced a recurrent Finsler connection which is neither h-metricalnor v-metrical, but is is recurrent with respect to both h-and v-covariant, derivatives. Suh a Finsler connection has been called an hv-recurrent Finsler connection.

To avoid confusions, we use h- and v-covariant derivatives with respect to Cartan's connection by |k *and* |k while these covariant derivatives with respect to generalized h-recurrent Finsler connection will be denoted by ∥ k and ∥ k respectively.

The quantities corresponding to generalized h-recurrent Finsler connection will be denoted by putting p on the top of the quantity followed by putting (a) in front of the quantity, while the quantities corresponding to Cartan's connection will be denoted as usual.

We have not used the raising and lowering of indices of objects, but if anywhere necessary, the metric g^{ij}will be used.

2. Fundamental Formulae

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x,y)$, where $x(=x^i)$ denotes the positional variable of F^n and y $(=y^i)$ denotes the components of a tangent vector with respect to x^i.

A Finsler connection of (M ^{n}, L) is triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of a V-connection F^i_{jk}a non-linear connection N^i_k and vertical connection C^i_{jk}(Matsumoto,1970). If a Finsler connection is given, the h- and covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j/k} = d_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk} \quad (2.1)$$

$$V^i_{j/k} = ð_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk}$$

$$where\ d_k = \dot{\partial}_k - N^m_K ð_m \quad and \quad ð_k = \frac{ð}{\partial x^k} \quad (2.2)$$

For any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ we have five torsion tensors and three curvature tensors, which are given by

(h) h-tensor	:	$T^i_{jk} = F^i_{jk} - F^i_{kj}$	(2.3)
(v) v-tensor	:	$S^i_{jk} = C^i_{jk} - C^i_{kj}$	(2.4)
(h) hv-tensor	:	$C^i_{jk} = as\ the\ connection C^i_{jk}$	(2.5)
(v) h-tensor	:	$R^i_{jk} = d_k N^i_j - d_j N^i_k$	(2.6)
(v) hv-tensor	:	$P^i_{jk} = \delta_k N^i_j - F^i_{kj}$	(2.7)
h-curvature	:	$R^i_{hjk} = d_k F^i_{hj} - d_j F^i_{hk} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj} + C^i_{hm} R^m_{jk}$	(2.8)
h-curvature	:	$P^i_{hjk} = ð_k F^i_{hj} - C^i_{hk\backslash j} + C^i_{hm} P^m_{jk}$	(2.9)
v-curvature	:	$S^i_{hjk} = ð_k C^i_{hj} - \partial_j C^i_{hk} + C^m_{hj} C^i_{mk} C^m_{hk} - C^i_{mj}$	(2.10)

3. Generalized *h*-Recurrent Finsler Connection

Let $p \neq 1$ be a real number. We define $\emptyset^{(P)}(x,y)$as

$$\emptyset^{(P)} = \frac{1}{p} L^p, (p \neq 0), \emptyset^{(0)} = \log L. \quad (3.1)$$

We denote

$$ð_i \emptyset^{(P)} and\ ð_i ð_j \emptyset^{(P)} \text{as} \emptyset^{(p)}_i \text{and } \emptyset^{(p)}_{ij}$$

and so on. Thus

$$\emptyset^{(p)}_i = L^{p-1} l_i, \emptyset^{(p)}_{ij} = L^{p-2}(g_{ij} + (p-2) l_i l_j) \quad (3.2)$$

In the following, we restrict our consideration to a domain, where the matrix $\|^{(p)}_{ij}\|$ is regular and then its inverse $\emptyset^{(p)ij}$is given by

$$\emptyset^{(p)ij} = L^{-(p-2)}\left[g^{ij} - \frac{(p-2)}{(p-1)} l^i l^j\right] \qquad (3.3)$$

Differentiating (3.2) w.r.t. y^k we have

$$\emptyset^{(p)}_{ijk} = L^{p-2}[2C_{ijk} + (p-2)L^{-1}\{h_{ij}l_k + h_{jkli} + h_{ki}l_j + (p-1)l_il_jl_k\}].$$

Let $a_k$be the component of a vector field, which is positively homogeneous of degree zero in y^i, then a finsler connection

$$\{F^{(p)i}_{jk}(a), N^{(p)i}_{k}(a), C^{(p)i}_{jk}(a)\}$$

will be called generalized *h*-recurrent Finsler connection, if *h*-covariant derivative of g_{ij} is recurrent and *v*-covariant derivative of $\emptyset^{(p)}_{ij}$ vanishes

i.e. $g_{ij\|\mathrm{k}} = a_k g_{ij} and\ \emptyset^{(p)}_{ij\|\mathrm{k}} = 0.$

To determine such a Finsler connection we have the following theorem.

Theorem 3.1:

Given a covariant vector field a_k, *there exists a unique Finsler connection* $\{F^{(p)i}_{jk}(a), N^{(p)i}_{k}(a), C^{(p)i}_{jk}(a)\}$ *satisfying the axioms*:

(c₁) $g_{ij\|\mathrm{k}} = a_k g_{ij}$

(c₂) $\emptyset^{(p)}_{ij\|k} = 0$

(c₃) *the deflection tensor factor* $D^{(p)i}_{k}(a)$ *vanishes i.e.,*$N^{(p)i}_{k}(a) = y^i F^{(p)i}_{jk}(a)$

(c₄) *the torsion tensor field* $T^{(p)i}_{jk}(a)$ *vanishes i.e.*$F^{(p)i}_{jk}(a) = F^{(p)i}_{kj}(a)$

(c₅) *the torsion tensor fold* $S^{(p)i}_{jk}(a)$ *vanishes i.e.*$C^{(p)i}_{jk}(a) = C^{(p)i}_{kj}(a)$.

Proof:

From (c₂) of Theorem 3.1, it follows that

$$\emptyset^{(p)}_{ij\|k} = \emptyset^{(p)}_{ijk} - C^{(p)}_{ijk}(a) - C^{(p)}_{jik} = 0 \qquad (3.4)$$

where

$$C^{(p)}_{ijk}(a) = \emptyset^{(p)}_{rj} C^{(p)}_{ik} r(a).$$

By cyclic interchanges of indices *i*, *j* and *k*, we get

$$C^{(p)}_{ijk}(a) = \frac{1}{2}(\emptyset^{(p)}_{ijk} + \emptyset^{(p)}_{jki} - \emptyset^{(p)}_{kij}) = \frac{1}{2}\emptyset^{(p)}_{ijk}$$

which implies

$$C^{(p)r}_{ik}(a) = \frac{1}{2}(\emptyset^{(p)}_{rj}\emptyset^{(p)}_{ijk} = C^r_{ik} + \sigma^{(p)r}_{ik} \qquad (3.5)$$

where $\sigma^{(p)r}_{ik}$ are as given below by (3.3)

$$\sigma^{(p)r}_{ik} = \frac{(p-2)}{2L}\left(\delta^r_i l_k + \delta^r_k l_i + \frac{h_{ik}}{p-1} l^r - l_i l_k l^r\right). \qquad (3.6)$$

From (c₁),(c₃) and (c₄) of Theorem 3.1 we have

$$F_{jk}^{(p)i}(a) = \Gamma_{jk}^{\cdot i} + Q_{jk}^{i} \tag{3.7}$$

and

$$N_{k}^{(i)i}(a) = a_{k}^{i\cdot} + T_{jk}^{i} \tag{3.8}$$

where

$$Q_{jk}^{i} = \frac{1}{2}\{a_0 C_{jk}^{i} + L^2(C_{jm}^{i}C_{k}^{m} + C_{km}^{i}C_{jk}^{m} - C_{jk}^{h}C_{m}^{i})$$
$$-(C_{j}^{i}y_k + C_{k}^{i}y_j - C_{jk}y^i) - (a_k\delta_j^i + a_j\delta_k^i - a^i g_{jk}) \tag{3.9}$$

$$T_{k}^{i} = \frac{1}{2}(a^i y_k + a_k y^i - a_0\delta_k^i - L^2 C_k^i) \tag{3.10}$$

and

$$C_k^i = C_{jk}^i a^j.$$

Thus the connection is uniquely determined.

Since the conditions (c_1), (c_3) and (c_4) of Theorem 3.1 are the same as the condition for the *hv*-recurrent Finsler connections, the v-connection and non-linear connection of *hv* Recurrent Finsler connection and generalized *h*-recurrent Finsler connection are identical.((Prasad et al., 2012)

for the case $p \neq 0$, we can replace the condition (c_1) of Theorem 3.1 by

$$\emptyset_{ij\|k}^{(p)} = \frac{1}{2}\, p\, a_k \emptyset_{ij}^{(p)}$$

and we have

Theorem 3.2: *A connection is uniquely determined for* $p \neq 0$, *by*

(c_1) $\emptyset_{ij\|k}^{(p)} = \frac{1}{2}\, p\, a_k \emptyset_{ij}^{(p)}$

(c_2) $\emptyset_{ij\|k}^{(p)} = 0$

(c_3) *The deflection tensor field* $D_{k}^{(p)i}(a) = 0;$ (3.11)

(c_4) *The torsion tensor field* $T_{jk}^{(p)i}(a) = 0;$

(c_5) *The deflection tensor field* $S_{jk}^{(p)i}(a) = 0;$

and is the same connection as that determined in Theorem 3.1.

Proof:

We take the connection $\{F_{jk}^{(p)i}(a), N_{k}^{(p)i}(a), C_{jk}^{(c)i}(a)\}$ determined in Theorem 3.1, then from the condition (c_1) of Theorem 3.1 we have

$$L_{\|k} = \frac{1}{2}a_k L \text{ and } l_{\|k} = \frac{1}{2}a_k l_i.$$

From (3.2), it is obvious that

$$\emptyset_{ij\|k}^{(p)} = \frac{1}{2}\, p\, a_k \emptyset_{ij}^{(p)}$$

Conversely, if the connection $\left\{F_{jk}^{(p)i}(a), N_k^{(p)i}(a), C_{jk}^{(p)i}(a)\right\}$ satisfies the conditions (3.11) (c_1) from homogeneity of $\emptyset_{ij}^{(p)}$ *and* $\emptyset_i^{(p)}$, we get

$$\emptyset_{i\|k}^{(p)} = \frac{1}{2}\, p\, a_k \emptyset_i^{(p)} \, and\, L_{\|k} = \frac{1}{2} L a_k.$$

Thus (3.2) leads to

$$l_{i\|k} = \frac{1}{2} a_k l_i \text{ and } g_{ij\|k} = a_k g_{ij}.$$

Uniqueness then follows from Theorem 3.1.

4. The Relation between Torsion and Curative Tensors of CΓ and Generalized *h*-Recurrent Finsler Connection

In the work (Prasad et al., 2014), the relation between (v) *h*-torsion tensor and (v) *hv* torsion tensors corresponding to *hv*-recurrent Finsler connection and Cartan's connection have been established. These are

$$R_{jk}^i(a,b) = R_{jk}^i + T_{j(k)}^i - T_{k(j)}^i + T_j^m \dot{\partial}_m T_k^i - T_k^m \dot{\partial}_m T_j^i \tag{4.1}$$

and

$$P_{jk}^i(a,b) = P_{jk}^i + \dot{\partial}_K T_j^i - Q_{kj}^i \tag{4.2}$$

When R_{jk}^i and P_{jk}^i are the (v) *h*-torsion and (v) *hv*-torsion tensor of Cartan's connection respectively and (k) denotes the h-covariant derivative with respect to Berwald's connection.

In view of (2.6) and (2.7) it follows that (v) *h*-torsion tensor and (v) *hv*-torsion tensor depends only upon v-connection and non-linear connection.

Thus (v) *h* torsion tensor $R_{jk}^{(p)i}$ (a) and (v) *hv*-torsion tensor $P_{jk}^{(p)i}$ (a) of generalized h-recurrent Finsler connection is also given by (4.1) and (4.2).

In view of (2.8), (3.7), (3.8) and (4.1) we have the following relations between *h*-curvature tensors of the generalized *h*-recurrent Finsler connection and Cartan's connection.

$$\begin{aligned} R_{hjk}^{(p)i}(a) = {} & R_{hjk}^i + Q_{hji\backslash k}^i - Q_{hk\backslash j}^i - T_k^m \dot{\partial}_m \Gamma_{hj}^{\cdot i} - T_k^m \dot{\partial}_m Q_{hj}^i \\ & + T_j^{m\cdot} \dot{\partial}_m \Gamma_{hk}^i + T_j^{m\cdot} \dot{\partial}_m Q_{hk}^i + Q_{hj}^m Q_{mk}^i - Q_{hk}^m Q_{mj}^i \\ & + \sigma_{hm}^{(p)i} R_{jk}^m + (C_{hm}^i + \sigma_{hm}^{(p)i})(T_{j(k)}^m - T_{k(j)}^m \quad + T_j^r \dot{\partial}_r T_k^m - Q_k^r \dot{\partial}_r T_j^m) \end{aligned}$$

The relation between hv-curvature tensor of generalized *h*-recurrent Finsler connection and Cartan's connection will be determined from (2.9), (3.7), (3.8) and (4.2). This relation is given by

$$P_{hjk}^{(p)i}(a) = P_{hjk}^i + T_j^m \dot{\partial}_m \left(C_{hk}^i + \sigma_{hk}^{(p)i}\right) + (C_{hm}^i + \sigma_{hm}^{(p)i}) \dot{\partial}_k T_j^m$$

$$+ Q^i_{hj}|_k - \sigma^{(p)i}_{hk|j} + \sigma^{(p)i}_{hm} P^m_{jk} + Q^i_{hm} C^m_{jk} - \sigma^{(p)m}_{hk} Q^i_{mj} + \sigma^{(p)i}_{mk} Q^m_{hj}. \quad (4.4)$$

The relation between *v*-curvature tensor of generalized *h*-recurrent Finsler connection and Cartan's connection will be determined from (2.10) and (3.5), which is given by.

$$S^{(p)i}_{hjk}(a) = S^i_{hjk} - \frac{(p-2)^2}{4\,(p-1)L^2}(h_{hj} h^i_k - h^i_j h_{hk}). \quad (4.5)$$

As the expressions (4.2), (4.4) and (4.5) are complicated to study the further properties of curvature tensors of generalized *h*-recurrent Finsler connections, in the next article we shall assume the particular form of the recurrence vector a_k.

5. A Particular From of Recurrence Vector a_k

If a_k be the vector along the unit vector l_k then from(3.10), (3.11) we get

$$T^i_k = -\frac{1}{2} L\delta^i_k, C^i_k = 0 \quad (5.1)$$

and

$$Q^i_{jk} = \frac{1}{2}(L\, C^i_{jk} - l_k C^i_k \delta^i_k + l^i g_{jk}) \quad (5.2)$$

Substituting the value of $T^i_j, Q^i_{jk} and\ \sigma^{(p)i}_{jk}$ from (5.1), (5.2) and (3.6) in the relation (4.3) we get

$$R^{(p)i}_{hjk}(\mathrm{a}) = R^i_{hjk} + \frac{1}{2}(P^i_{hji} - P^i_{hkj}) + \frac{1}{4} L^2 S^i_{hjk}$$

$$+ \frac{1}{4}(\delta^i_k g_{hj} - \delta^i_k g_{hk}) + \frac{(p-2)}{2L}\left(R^i_{jk} l_h - \frac{l^i}{p-1} - R_{hjk}\right)$$

$$+ \frac{(p-2)}{8}\left\{\delta^i_k l_h l_j - \delta^i_j l_h l_k + \frac{1}{p-1}(g_{hk} l^i l_j - g_{hj} l^i l_k)\right\} \quad (5.3)$$

A Finsler space of scalar curvature *K* is defined by Matsumoto (Matsumoto, 2001, Matsumoto, 1998) which is characterized by

$$R^i_{hjk} = K\,(\delta^i_k g_{hj} - \delta^i_j g_{hk}) \quad (5.4)$$

From which, we get

$$R^i_{jk} = K\,(\delta^i_k y_j - \delta^i_j y_k) \quad (5.5)$$

If the scalar curvature *K* is constant, then F_n is called a Finsler space of constant curvature.

Since $P^i_{hjk} - P^i_{hkj} = -S^i_{hjk|0}$.

Substituting (5.4) and (5.5) in (5.3) we get

$$R^{(p)i}_{hjk}(a) = \frac{1}{4}\mathrm{L}^2 S^i_{hjk} - \frac{1}{2} S^i_{hjk|0} + \left(K + \frac{1}{4}\right)\left[\delta^i_k g_{hj} - \delta^i_j g_{hk}\right)$$

$$+ \frac{(\mathrm{p}-2)}{2}\left\{\left(\delta^i_k l_j l_h - \delta^i_k l_k l_h\right) + \frac{1}{(p-1)}\,(g_{hk} l_j l^i - g_{hj} l_k l^i)\right\}\Big] \quad (5.6)$$

This relation gives the following theorem:

Theorem 5.1: If the h-curvature tensor of generalized h-recurrent Finsler connection with respect to the recurrent vector L_k vanishes and (F^n, L) is of constant curvature $\left(-\frac{1}{4}\right)$, then

$$S^i_{hjk|0} + \frac{1}{2} L\, S^i_{hjk}.$$

Substituting (5.1) and (5.2)in (4.4) and using (2.1) we get

$$P^{(p)i}_{hjk}(a) = P^i_{hjk} + \frac{1}{2}\mathrm{L}\, S^i_{hjk} + \frac{1}{2}(l^i C_{jkh} - l_h C^i_{jk} - l_k C^i_{hj}) + \frac{1}{2\mathrm{L}}(h_k g_{hj} - h_{hk}\delta^i_j - h_{jk}\delta^i_h)$$

$$-\frac{1}{2}\mathrm{L}(\sigma^{(p)i}_{hm}|j - \frac{1}{2}\mathrm{L}(\sigma^{(p)i}_{hm} C^m_{kj}) - \sigma^{(p)i}_{hm|j} - \frac{1}{2} l_k \sigma^{(p)i}_{jh} - \sigma^{(p)i}_{hjk} l^i - \sigma^{(p)i}_{jk} l_h$$

$$+\frac{(\mathrm{p}-2)}{2\mathrm{L}}\left(P^i_{jk} l_h + \frac{1}{\mathrm{p}-1}\mathrm{P_{hjk}}\mathrm{l^i}\right) + \frac{(\mathrm{p}-2)}{4\mathrm{L}}\mathrm{g_{hj}}\delta^i_k + \frac{(\mathrm{p}-2)}{4(\mathrm{p}-2)\mathrm{L}}\delta^i_j h_{hk}. \qquad (5.7)$$

Conclusion

a) Given a covariant vector field a_k, there exists a unique Finsler connection $\{F^{(p)i}_{jk}(a), N^{(p)i}_k(a), C^{(p)i}_{jk}(a)\}$ satisfying the following axioms:

- (c_1) $g_{ij\|\mathrm{k}} = a_k g_{ij}$
- (c_2) $\emptyset^{(p)}_{ij\|k} = 0$
- (c3) the deflection tensor factor $\mathrm{D^{(p)i}_k}$(a) vanishes i.e.,$\mathrm{N^{(p)i}_k}$(a) =$\mathrm{y^i F^{(p)i}_{jk}}$(a)
- (c_4) the torsion tensor field $T^{(p)i}_{jk}(a)$ vanishes i.e.$F^{(p)i}_{jk}(a) = F^{(p)i}_{kj}(a)$
- (c_5) the torsion tensor field $S^{(p)i}_{jk}(a)$ vanishes i.e.$C^{(p)i}_{jk}(a) = C^{(p)i}_{kj}(a)$

b) A connection is uniquely determined for $p \neq 0$, by

- (c_1) $\emptyset^{(p)}_{ij\|k} = \frac{1}{2}\, p\, a_k \emptyset^{(p)}_{ij}$
- (c_2) $\emptyset^{(p)}_{ij\|k} = 0$
- (c_3) The deflection tensor field $D^{(p)i}_k(a) = 0;$
- (c_4) The torsion tensor field $T^{(p)i}_{jk}(a) = 0;$
- (c_5) The deflection tensor field $S^{(p)i}_{jk}(a) = 0;$

and is the same connection as that determined in Theorem 3.1.

c) If the h-curvature tensor of generalized h-recurrent Finsler connection with respect to the recurrent vector L_k vanishes and (F^n, L) is of constant curvature $\left(-\frac{1}{4}\right)$, then
$S^i_{hjk|0} + \frac{1}{2} L\, S^i_{hjk}$.

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