A Note on Natural Transformation of Exton's Triple Series Hypergeometric Function

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Abstract: *Triple series hypergeometric functions are very important from the applications point of view. Exton had defined 20 triple hypergeometric functions namely X_1 , X_2 , ... , X_{20}. Integral transform technique is widely using for research purpose. Natural transform is a new kind of integral transform and generalization of Laplace transform. In the present research note, we give the Natural integrals of the triple series hypergeometric function due to Exton.*

Keywords: *Hypergeometric functions, Exoton's triple series, N-transform.*

1. Introduction

Triple series hypergeometric functions are well known in literature and large number of applications are available. Srivastava and Kashyap **[11]** present several interesting applications of hypergeometric series in one and more variable in queuing theory and stochastic process. Lauricella **[8]** introduced 14 complete set of hypergeometric series in three variables. Saran **[9]** initiated systematic study of Lauricella set. Srivastava and Karlsson **[10]** give detailed account on triple series in his famous monograph. In 1982 Exton **[2]** had defined 20 triple functions namely X_1, X_2 , ... ,X_{20} in his famous note. In the same note, Laplace integrals of Exoton's series **[7]** and wide applications are also given which shows importance and applicability of the functions. Integral Transform technique is an important and frequent using tool in mathematical and computational research. Recently developed Natural transform is widely using for application purpose. The natural transformation **[3]**, initially defined by khan and khan as N-transform, who studied properties and applications. Belgacem *et. al.* **[6],[1]** defined its inverse and studied some additional fundamental properties of this transform and named it the Natural transform.

2. Results Required

The Natural transform **[5],[6]** $R(s , u)$ of the function for all $t \geq 0$, is given by

$$N^+\left[f(t)\right] = R(s,u) = \frac{1}{u}\int_0^\infty e^{-st} f(ut)dt \qquad s > 0, u > 0$$

$$R(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)dt, \tag{2.1}$$

where t, u are time variables and s is the frequency variable, provided the function $f(t)$ is defined in a set

$$A = \left\{ f(t) : \exists \, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}} ; t \in (-1)^j \times [0, \infty) \right\}, \quad (2.2)$$

where M is finite constant, τ_1, τ_2 may be finite or infinite. The discrete form of natural transform [4] is given by

$$N^+[f(t)] = R(s, u) = \sum_0^\infty \frac{n! a_n u^n}{s^{n+1}} \quad (2.3)$$

The inverse Natural transformation is defined by

$$N^{-1}[R(s, u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s, u) ds \quad (2.4)$$

By using these definitions, shifted factorial is defined as follows:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \frac{s}{u} \int_0^\infty e^{-\frac{st}{u}} \left(\frac{st}{u} \right)^{a+m-1} dt$$

Exton has defined following 20 triple series hypergeometric functions:

$$X_1(a, b; c, d; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c)_m (d)_{n+p} m! n! p!} \quad (2.5)$$

$$X_2(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!} \quad (2.6)$$

$$X_3(a, b; c, d; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+n+p}(b)_{n+p} x^m y^n z^p}{(c)_{m+n} (d)_p m! n! p!} \quad (2.7)$$

$$X_4(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+n+p}(b)_{n+p} x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!} \quad (2.8)$$

$$X_5(a, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_{m+n+p} m! n! p!} \quad (2.9)$$

$$X_6(a, b_1, b_2; c, d; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_{m+n} (d)_p m! n! p!} \quad (2.10)$$

$$X_7(a, b_1, b_2; c, d; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_m (d)_{n+p} m! n! p!} \quad (2.11)$$

$$X_8\left(a,b_1,b_2;c_1,c_2,c_3;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n(b_2)_p\, x^m y^n z^p}{(c_1)_m(c_2)_n(c_3)_p\, m!n!p!} \tag{2.12}$$

$$X_9\left(a,b;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}\, x^m y^n z^p}{(c)_{m+n+p}\, m!n!p!} \tag{2.13}$$

$$X_{10}\left(a,b;c,d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}\, x^m y^n z^p}{(c)_{m+n}(d)_p\, m!n!p!} \tag{2.14}$$

$$X_{11}\left(a,b;c,d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}\, x^m y^n z^p}{(c)_{m+p}(d)_n\, m!n!p!} \tag{2.15}$$

$$X_{12}\left(a,b;c_1,c_2,c_3;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}\, x^m y^n z^p}{(c_1)_m(c_2)_n(c_3)_p\, m!n!p!} \tag{2.16}$$

$$X_{13}\left(a,b,c;d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p\, x^m y^n z^p}{(d)_{m+n+p}\, m!n!p!} \tag{2.17}$$

$$X_{14}\left(a,b,c;d,d';x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p\, x^m y^n z^p}{(d)_{m+n}(d')_p\, m!n!p!} \tag{2.18}$$

$$X_{15}\left(a,b,c;d,d';x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p\, x^m y^n z^p}{(d)_m(d')_{n+p}\, m!n!p!} \tag{2.19}$$

$$X_{16}\left(a,b,c;d,d';x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p\, x^m y^n z^p}{(d)_{m+p}(d')_n\, m!n!p!} \tag{2.20}$$

$$X_{17}\left(a,b,c;d_1,d_2,d_3;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p\, x^m y^n z^p}{(d_1)_m(d_2)_n(d_3)_p\, m!n!p!} \tag{2.21}$$

$$X_{18}\left(a,b,b',c;d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p\, x^m y^n z^p}{(d)_{m+n+p}\, m!n!p!} \tag{2.22}$$

$$X_{19}\left(a,b,b',c;d,d';x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p\, x^m y^n z^p}{(d)_m(d')_{n+p}\, m!n!p!} \tag{2.23}$$

$$X_{20}\left(a,b,b',c;d,d';x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p\, x^m y^n z^p}{(d)_{m+p}(d')_n\, m!n!p!} \tag{2.24}$$

3. Main Results

The following twenty interesting integrals involving generalized hypergeometric function will be evaluated in this paper:

1. $X_1(a,b;c,d;x,y,z)=$

$$\frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}{}_0F_1\left(-;c;\frac{xs^2}{u^2}\right){}_0F_1\left(-;d;ys^2uv+zstu^2\right)dsdt \quad \textbf{(3.1)}$$

2. $X_2(a,b;c_1,c_2,c_3;x,y,z)=\dfrac{1}{\Gamma(a)}\dfrac{1}{u}\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\dfrac{s}{u}\right)^{a-1}{}_0F_1\left(-;c_1;\dfrac{xs^2}{u^2}\right){}_0F_1\left(-;c_2;\dfrac{ys^2}{u^2}\right){}_1F_1(b;c_3;\dfrac{zs}{u})ds \quad \textbf{(3.2)}$

3. $X_3(a,b;c,d;x,y,z)=$

$$\frac{1}{\Gamma(a)}\frac{1}{\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}{}_0F_1\left(-;c;\frac{xs^2uv+ytu^2s}{u^2}\right)\left(\frac{t}{v}\right)^{b-1}{}_0F_1\left(-;d;\frac{zst}{uv}\right)dsdt \quad \textbf{(3.3)}$$

4. $X_4(a,b;c_1,c_2,c_3;x,y,z)=\dfrac{1}{\Gamma(a)}\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\dfrac{s}{u}\right)^{a-1}{}_0F_1\left(-;c_1;x\dfrac{s^2}{u^2}\right)\psi_2\left(b;c_2,c_3;y\dfrac{s}{u},z\dfrac{s}{u}\right) \quad \textbf{(3.4)}$

5. $X_5(a,b_1,b_2;c;x,y,z)=$

$$\frac{1}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)}\frac{1}{uvw}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t_1}{v}-\frac{t_2}{w}\right)}\left(-\frac{s}{u}\right)^{a-1}\left(\frac{t_1}{v}\right)^{b_1-1}\left(\frac{t_2}{w}\right)^{b_2-1}{}_0F_1\left(-;c;x\left(\frac{s}{u}\right)^2+y\left(\frac{st_1}{uv}\right)+z\left(\frac{st_2}{uw}\right)\right)dsdt_1dt_2 \quad \textbf{(3.5)}$$

6. $X_6(a,b_1,b_2;c,d;x,y,z)=\dfrac{1}{\Gamma(a)\Gamma(b_1)}\dfrac{1}{uv}$

$$\times\int_0^\infty\int_0^\infty e^{\left(\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b_1-1}{}_0F_1\left(-;c;\frac{xs^2}{u^2}+\frac{yst}{uv}\right){}_1F_1\left(b_2;d;z\frac{s}{u}\right)dsdt \quad \textbf{(3.6)}$$

7. $X_7(a,b_1,b_2;c,d;x,y,z)=\dfrac{1}{u\Gamma(a)}\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\dfrac{s}{u}\right)^{a-1}\phi_2\left(b_1,b_2;d;y\dfrac{s}{u},z\dfrac{s}{u}\right){}_0F_1\left(-;c;x\dfrac{s^2}{u^2}\right)ds \quad \textbf{(3.7)}$

8. $X_8(a,b_1,b_2;c_1,c_2,c_3;x,y,z)=\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\dfrac{s}{u}\right)^{a-1}{}_0F_1\left(-;c_1;x\left(\dfrac{s}{u}\right)^2\right){}_1F_1\left(b_1;c_2;y\dfrac{s}{u}\right){}_1F_1(b_2;c_3;z\dfrac{s}{u})ds \quad \textbf{(3.8)}$

9. $X_9(a,b;c;x,y,z)=\dfrac{1}{\Gamma(a)\Gamma(b)}\dfrac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(\frac{s}{u}-\frac{t}{v}\right)}\left(\dfrac{s}{u}\right)^{a-1}\left(\dfrac{t}{v}\right)^{b-1}\sum_{m=0}^\infty\dfrac{1}{(c)_m}\dfrac{\left(x\dfrac{s^2}{u^2}+y\dfrac{st}{u^2}+z\dfrac{t^2}{uv}\right)^m}{m!}dsdt \quad \textbf{(3.9)}$

10. $X_{10}(a,b;c,d;x,y,z) =$

$$\frac{1}{uv}\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\frac{s}{u}\right)^{a-1}e^{\left(-\frac{t}{v}\right)}\left(\frac{t}{v}\right)^{b-1}\,_0F_1\left(-;c;x\left(\frac{s}{u}\right)^2+yst\right)\,_0F_1\left(-;d;z\left(\frac{t}{v}\right)^2\right)dsdt \qquad \textbf{(3.10)}$$

11. $X_{11}(a,b;c,d;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\,_0F_1\left(-;c;xs^2+zt^2\right)\,_0F_1\left(-;d;zst\right)dsdt$ **(3.11)**

12. $X_{12}\left(a,b;c_1,c_2,c_3;x,y,z\right)=$

$$\frac{1}{uv}\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\,_0F_1\left(-;c_1;x\frac{s^2}{u^2}\right)\,_0F_1\left(-;c_2;y\frac{st}{uv}\right)\,_0F_1\left(-;c_3;z\frac{t^2}{v^2}\right)dsdt \quad \textbf{(3.12)}$$

13. $X_{13}(a,b,c;d;x,y,z)=$

$$\frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)}\frac{1}{uvw}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(-\frac{s_1}{u}-\frac{s_2}{v}-\frac{s_3}{w}\right)}\left(-\frac{s_1}{u}\right)^{a-1}\left(\frac{s_2}{v}\right)^{b-1}\left(\frac{s_3}{w}\right)^{c-1}\,_0F_1\left(-;d;\frac{xs_1^2wv+yus_1s_2w+zu^2s_2s_3}{u^2wv}\right)ds_1ds_2ds_3 \quad \textbf{(3.13)}$$

14. $X_{14}\left(a,b,c;d,d';x,y,z\right)=$

$$\frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\,_0F_1\left(-;d;\frac{xs^2v+ysut}{uv}\right)\,_1F_1\left(c;d';\frac{zt}{v}\right)dsdt \qquad \textbf{(3.14)}$$

15. $X_{15}\left(a,b,c;d,d';x,y,z\right)=$

$$\frac{1}{uv}\frac{1}{\Gamma(a)\Gamma(c)}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{c-1}\,_0F_1\left(-;d;x\left(\frac{s}{u}\right)^2\right)\,_1F_1\left(b;d';\frac{ysv+zut}{uv}\right)dsdt \qquad \textbf{(3.15)}$$

16. $X_{16}\left(a,b,c;d,d';x,y,z\right)=$

$$\frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)}\frac{1}{uvw}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(-\frac{s_1}{u}-\frac{s_2}{v}-\frac{s_3}{w}\right)}\left(\frac{s_1}{u}\right)^{a-1}\left(\frac{s_2}{v}\right)^{b-1}\left(\frac{s_3}{w}\right)^{c-1}\,_0F_1\left(-;d;\frac{xs_1^2vw+zu^2s_2s_3}{u^2vw}\right)\,_0F_1\left(-;d';y\frac{s_1s_2}{uv}\right)ds_1ds_2ds_3 \quad \textbf{(3.16)}$$

17. $X_{17}\left(a,b,c;d_1,d_2,d_3;x,y,z\right)=$

$$\frac{1}{\Gamma(a)\Gamma(c)}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{c-1}\,_0F_1\left(-;d_1;x\frac{s^2}{u^2}\right)\Psi_2\left(b;d_2,d_3;y\frac{s}{u},z\frac{t}{v}\right)dsdt \qquad \textbf{(3.17)}$$

18. $X_{18}\left(a,b,b',c;d;x,y,z\right)=$

$$\frac{1}{uvwt}\frac{1}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(-\frac{s_1}{u}-\frac{s_2}{v}-\frac{s_3}{w}-\frac{s_4}{t}\right)}\left(\frac{s_1}{u}\right)^{a-1}\left(\frac{s_2}{v}\right)^{b-1}\left(\frac{s_3}{w}\right)^{b'-1}\left(\frac{s_4}{t}\right)^{c-1}\,_0F_1\left(-;d;\frac{xs_1^2wvt+ys_1s_2uwt+zu^2vs_3s_4}{u^2wvt}\right)ds_1ds_2ds_3ds_4 \quad \textbf{(3.18)}$$

19. $X_{19}\left(a,b,b^{'},c;d,d^{'};x,y,z\right)=$

$$\frac{1}{\Gamma(a)\Gamma(c)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{c-1}{}_0F_1\left(-;d_1;x\frac{s^2}{u^2}\right)\phi_2\left(b,b^{'};d^{'};\frac{ys}{u},\frac{zt}{v}\right)dsdt \quad \textbf{(3.19)}$$

20. $X_{20}\left(a,b,b^{'},c;d,d^{'};x,y,z\right)=$

$$\frac{1}{\Gamma(a)\Gamma\left(b^{'}\right)\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(-\frac{s_1}{u}-\frac{s_2}{v}-\frac{s_3}{w}\right)}\left(\frac{s_1}{u}\right)^{a-1}\left(\frac{s_2}{v}\right)^{b^{'}-1}\left(\frac{s_3}{w}\right)^{c-1}{}_0F_1\left(-;d;\frac{xs_1^2vw+zu^2s_2s_3}{u^2vw}\right){}_1F_1\left(b;d^{'};\frac{ys_1}{u}\right)ds_1ds_2ds_3$$

$$\textbf{(3.20)}$$

4. Derivation

In order to derive the main results, first we shall derive the result (3.1), i.e.

$$X_1\left(a,b;c,d;x,y,z\right)$$

$$=\frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}{}_0F_1\left(-;c;\frac{xs^2}{u^2}\right){}_0F_1\left(-;d;\frac{ys^2}{u^2}+\frac{zst}{uv}\right)dsdt \quad \text{(4.1)}$$

Proof:

To derive (4.1), we consider L.H.S as I

$$I=X_1\left(a,b;c,d;x,y,z\right)=\sum_{m,n,p=0}^\infty\frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c)_m(d)_{n+p}m!n!p!}$$

$$=\frac{1}{\Gamma(a)\Gamma(b)}\sum_{m,n,p=0}^\infty\frac{x^m y^n z^p}{(c)_m(d)_{n+p}m!n!p!}\int_0^\infty e^{\left(-\frac{s}{u}\right)}\left(\frac{s}{u}\right)^{2m+2n+p+a-1}\left(\frac{ds}{u}\right)\int_0^\infty e^{\left(-\frac{t}{v}\right)}\left(\frac{t}{v}\right)^{p+b-1}\left(\frac{dt}{v}\right)$$

Put $n=n-p$, we get

$$I=\frac{1}{\Gamma(a)\Gamma(b)}\sum_{m,n=0}^\infty\sum_{p=0}^n\frac{x^m y^{n-p}z^p}{(c)_m(d)_{n-p+p}m!(n-p)!p!}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{2m+2n-2p+p+a-1}\left(\frac{t}{v}\right)^{b+p-1}\frac{ds}{u}\frac{dt}{v}$$

$$=\frac{1}{\Gamma(a)\Gamma(b)}\sum_{m,n=0}^\infty\sum_{p=0}^n\frac{x^m y^{n-p}z^p}{(c)_m(d)_n m!(n-p)!p!}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{2m+2n-p+a-1}\left(\frac{t}{v}\right)^{b+p-1}\frac{ds}{u}\frac{dt}{v}$$

Using the formula $\dfrac{1}{(n-p)!} = \dfrac{(-n)_p (-1)^p}{n!}$ we have

$$I = \frac{1}{\Gamma(a)\Gamma(b)}\sum_{p=0}^{n}\sum_{m,n=0}^{\infty}\frac{x^m y^{n-p}z^p(-n)_p(-1)^p}{(c)_m(d)_n m!n!p!}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{2m+2n+a-1}\left(\frac{t}{v}\right)^{b-1}\left(\frac{tu}{sv}\right)^p\left(\frac{ds}{u}\right)\left(\frac{dt}{v}\right)$$

$$= \frac{1}{\Gamma(a)\Gamma(b)}\sum_{p=0}^{n}\sum_{m,n=0}^{\infty}\frac{x^m y^{n}(-n)_p(-1)^p}{(c)_m(d)_n m!n!p!}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1+2m+2n}\left(\frac{t}{v}\right)^{b-1}\left(\frac{ztu}{ysv}\right)^p dsdt$$

$$\sum_{p=0}^{n}\frac{(-n)_p(-1)^p}{p!}\left(\frac{ztu}{ysv}\right)^p = \left(1+\frac{ztu}{ysv}\right)^n = \left(\frac{ysv+ztu}{ysv}\right)^n$$

Changing the order of series and integration, we get

$$I = \frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\sum_{m,n=0}^{\infty}\frac{x^m y^n}{(c)_m(d)_n m!n!}\left(\frac{s}{u}\right)^{2m}\left(\frac{ysv+ztu}{ysv}\right)^n\left(\frac{s}{u}\right)^{2n} dsdt$$

$$= \frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\sum_{m,n=0}^{\infty}\frac{x^m}{(c)_m(d)_n m!n!}\left(\frac{s^2}{u^2}\right)^m\left(\frac{ysv+ztu}{sv}\right)^n\left(\frac{s^2}{u^2}\right)^n dsdt$$

$$= \frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}\sum_{m=0}^{\infty}\frac{1}{(c)_m m!}\left(\frac{xs^2}{u^2}\right)^m\sum_{n=0}^{\infty}\frac{1}{(d)_n n!}\left(\frac{ys^2v+zstu}{u^2v}\right)^n dsdt$$

$$= \frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{\left(-\frac{s}{u}-\frac{t}{v}\right)}\left(\frac{s}{u}\right)^{a-1}\left(\frac{t}{v}\right)^{b-1}{}_0F_1\left(-;c;\frac{xs^2}{u^2}\right){}_0F_1\left(-;d;\frac{ys^2v+zstu}{u^2v}\right) dsdt$$

$= $ R.H.S

Special Case:

If we put $u = 1$ and $v = 1$ in (4.1) then we get

$$X_1\left(a,b;c,d;x,y,z\right)$$
$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t} s^{a-1} t^{b-1} {}_0F_1\left(-;c;xs^2\right) {}_0F_1\left(-;d;ys^2+zst\right) ds\, dt$$

(4.2)

This verifies the Exton result. Rest could be verified in the same manner.

Acknowledgements

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