On the Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series

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Abstract: *In this paper, we have proved the degree of approximation of function belonging to* $L[0,\infty)$ *by Nörlund Summability of Fourier-Laguerre series at the end point* $x = 0$. *The purpose of this paper is to concentrate on the approximation relations of the function in* $L[0,\infty)$ *by Nörlund Summability of Fourier-Laguerre series associate with the given function motivated by the works* **[3]**, **[9]** *and* **[13]**.

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1. Introduction

The concepts of product summability methods are more powerful than the individual summability methods and it gives an approximation for wider class of functions than the individual methods **[6]**. A bulk number of researchers have studied the degree of approximation of a function using different summability means of its Fourier-Laguerre series **[1]**, **[5]**, **[7]**, **[10]**, **[11]**, **[14]** and **[15]**. This can be done at the point after replacing the continuity condition in Szegö theorem by much lighter conditions.

Let $f(t)$ be a Lebesgue measurable function in the interval $(0, \infty)$ such that the integral

$$\int_0^\infty e^{-x}\, x^\alpha f(x)\, L_n^{(\alpha)}(x)dx,\ \ \alpha > -1 \tag{1.1}$$

exists, where $L_n^{(\alpha)}(x)$ is the n^{th} Laguerre polynomial of order $\alpha > -1$ defined the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, \omega^n = (1-\omega)^{-\alpha-1}\, e^{\frac{-x\omega}{1-\omega}} \tag{1.2}$$

The Fourier series of the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \qquad \alpha > -1, \tag{1.3}$$

where the coefficients a_n are the defined by the following formulae

$$\Gamma(\alpha+1)\, A_n^\alpha\ \ a_n = \int_0^\infty e^{-x}\, x^\alpha\, f(x)\, L_n^{(\alpha)}(x)dx \tag{1.4}$$

where $A_n^\alpha = \binom{n+a}{n} \sim n^\alpha.$

Also, we write $\phi(x) = \frac{1}{\Gamma(\alpha+1)} e^{-x} x^\alpha$ (1.5)

Let $\sum u_n$ be a given infinite series with the sequence $\{S_n\}$ of its partial sums. Let $\{p_n\}$ be a sequence of real or complex constants with p_n as its non-vanishing n^{th} partial sum. The sequence to sequence transformation is given by

$$\tau_n = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} S_m. \quad (1.6)$$

which defines the sequence of Nӧrlund means of the sequence $\{S_n\}$, generated by the sequence of coefficients $\{p_n\}$. If $\tau_n \to s$ then $\sum u_n$ or the sequence $\{S_n\}$ to the sum *S*. If the sequence of Nӧrlund mean $\{\tau_n\}$ is of bounded variation i.e. if

$$|\tau_n - \tau_{n-1}| < \infty \quad (1.7)$$

then the series $\sum u_n$is said to be absolute Nӧrlund summable **[5], [6]** and **[12]**. The absolute Nӧrlund summability of a Fourier series has been studied by several researchers like Bor **[5]**, **[7]**, Mazhar **[10]**, Fadden **[11]**, Padhy, Tripathi and Mishra **[14]** and Siddiqui, Gi, Bohar and Brono **[15]**.

In 1976, Yadav **[19]** was the first to establish a result on absolute Nӧrlund summability of Laguerre series at origin and in 1979, Beohar and Jadiya **[2]** established a result on the absolute Nӧrlund summability of Laguerre series at the point $x = 0$. Later on, several researchers like Alghamdi & Mursaleen **[1]**, Beohar and Jadiya **[2], [3]**, Khatri and Mishra **[8]**, Karasniqi **[9]**, Nigam and Sharma **[13]**, Shanker **[16]**, Tiwari and Kachhara **[18]** obtained the degree of approximations of L[0,∞) of the Fourier-Laguerre series by Cesaro mean Nӧrlund, Euler, (C,1)(E,q),(C,2)(E,q) harmonic-Euler mean et al. Concerning the absolute Nӧrlund summability of Laguerre series Yadav **[19]** has proved the following theorem:

Theorem 1.1: If $\{p_n\}$ be a non-negative and non-increasing sequence of constant such that

$P_n = \sum_{r=0}^{n} p_r \to \infty$ and $\sum_n \frac{n^{\frac{(-1+2\alpha)}{4}}}{P_n}$ is convergent. i.e. $\sum_n \frac{n^{\frac{(-1+2\alpha)}{4}}}{P_n} < \infty$, then the Fourier series (3) is $|N, p_n|$ summable at the end point $x = 0$, provided that for some small positive ϵ,

$$\phi(x) \epsilon BV[0, \infty) \quad (1.8)$$

where

$$\int_n^\infty e^{\frac{-x}{2}} x^{\frac{\alpha}{2} - \frac{7}{12}} |f(x)| dx = o\left(n^{\frac{-1}{2}}\right) \text{ and } F(x) \text{ is bounded in } [0,\epsilon]. \quad (1.9)$$

Where,

$$\emptyset(x) = [f(x) - f(0)]. \frac{e^{-x} .x^\alpha}{\Gamma(\alpha+1)} \quad (1.10)$$

and,

$$F(x) = \frac{[f(x) - f(0)]. e^{-x}. x^{\frac{(-1+2\alpha)}{4}}}{\Gamma(\alpha+1)} \quad (1.11)$$

2. Main Results

Before proceeding with the main result, we begin with recalling some Lemmas that are required to prove main theorem in this paper.

Lemma 2.1: (Szego [17], p.177) Let α be arbitrary and real, c and w be fixed positive constants, and let $n \to \infty$. Then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\frac{\alpha}{2}-\frac{1}{4}}.o\left(n^{\frac{2\alpha-1}{4}}\right), if \ \frac{c}{n} \le x \le w \\ o(n^{\alpha}), \quad if \ 0 \le x \le \frac{c}{n} \end{cases} \tag{2.1}$$

Lemma 2.2: (Szegö[17], p. 240) Let α be arbitrary and real, $w > 0, \ 0 < \eta < 4$.Then we have for $n \to \infty$,

$$max.e^{\frac{-x}{2}}.x^{\frac{2\alpha+1}{4}}.\left|L_n^{(\alpha)}(x)\right| \sim \begin{cases} n^{\frac{\alpha}{2}-\frac{1}{4}}, \ if \ w \le x \le (4-\eta).n \\ n^{\frac{\alpha}{2}-\frac{1}{12}}, \ if \ x \ge w \end{cases} \tag{2.2}$$

(2.3)

Lemma 2.3: (Bhatta [4]) Let $\sum a_n$ be an infinite series with S_n as its n^{th} partial sum and $\{p_n\}$ be a non-increasing sequence such that $p_n \to \infty$. If $\sum_n \frac{|s_n|}{P_n} < \infty$ i.e. convergent, then the series $\sum_n a_n$ is $|N, P_n|$ summable.

Considering the above facts, we prove the following theorem:

Theorem 2.4: Let $\chi(x)$ be a non-negative, and non-increasing function of x such that $x^{\alpha}.\chi\left(\frac{1}{x}\right) \to 0$ as $n \to 0$. Let $\{p_n\}$ be a non-negative and monotonic non-increasing sequence of constants with P_n as its non-vanishing n^{th} partial sum such that $\sum_n \frac{|\chi(n)|}{P_n} < \infty$ i.e. convergent. Let $\frac{-1}{2} > \alpha > -1$ and w is a fixed positive constant. If

$$\int_x^w \left|\frac{\phi(s)}{S^{\alpha+1}}\right| ds = o\left[\chi\left(\frac{1}{x}\right)\right], \text{ as } x \to 0, \tag{2.4}$$

$$\int_w^n e^{\frac{x}{2}}.x^{\frac{-\alpha}{2}-\frac{3}{4}}|\phi(x)|dx = o\left[n^{\frac{-\alpha}{2}-\frac{1}{4}}.\ \chi(n)\right] \tag{2.5}$$

and

$$\int_n^{\infty} e^{\frac{x}{2}}.x^{\frac{-(6\alpha+7)}{12}}|\phi(x)|dx = o\left[n^{\frac{-(2\alpha+1)}{4}}.\ \chi(n)\right] \text{ as } n \to \infty, \tag{2.6}$$

then the Fourier-Laguerre series (1.3) is $|N, p_n|$ summable at the point $x = 0$.

Proof:

First of all , under the hypothesis (2.4), we prove

$$\int_0^x |\phi(s)|\,dx = o\left[x^{\alpha+1}.\chi\left(\frac{1}{x}\right)\right] \tag{2.7}$$

For this, we have $\quad I(s) = \int_x^w \frac{|\phi(s)|}{S^{\alpha+1}} ds = o\left[\chi\left(\frac{1}{x}\right)\right]$

Then $\quad |\phi(s)| = -I'(s).\ S^{\alpha+1}$

Therefore, $\quad \int_0^x |\phi(s)|\,ds = -\int_0^x S^{\alpha+1}.\ I'(s)ds$

$$= -[S^{\alpha+1}\ I(S)]_0^x + (\alpha+1)\int_0^x S^{\alpha}.I(s)ds$$

$$= o\left[S^{\alpha+1}.\chi\left(\frac{1}{s}\right)\right] + o\left[S^{\alpha+1}.\chi\left(\frac{1}{s}\right)\right]$$
$$= o\left[S^{\alpha+1}.\chi\left(\frac{1}{s}\right)\right]$$

Then the proof of the theorem is as follows:

$$\text{Since} \quad L_n^{(\alpha)}(0) = \binom{n+\alpha}{\alpha} \tag{2.8}$$

therefore $S_n(0) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0)$

$$= \sum_{k=0}^{n} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-x}.\ x^\alpha f(x) \sum_{k=0}^{n} L_k^{(\alpha)}(x)dx$$
$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-x}.x^\alpha f(x) L_n^{(\alpha+1)}(x)dx$$

Again, due to the orthogonality of Laguerre polynomials, we have

$$S_n(0) - f(0) = \int_0^\infty \phi(x)\, L_n^{(\alpha+1)}(x)dx$$
$$=\left(\int_0^{\frac{c}{n}} + \int_{\frac{c}{n}}^{w} + \int_w^n + \int_n^\delta\right) \phi(x) L_n^{(\alpha+1)}(x)dx$$
$$= I_1 + I_2 + I_3 + I_4 \text{ (say)} \tag{2.9}$$

First we consider I_1,

$$|I_1| = \int_0^{\frac{c}{n}} |\phi(x)|\, L_n^{\alpha+1}(x)dx$$
$$= \text{o}(n^{\alpha+1}) \int_0^{\frac{c}{n}} |\phi(x)|dx$$
$$= o[\chi(n)], \quad \text{as } n \to \infty. \tag{2.10}$$

Next we consider I_2, using (2.1) and (2.4), we have

$$|I_2| = o\left(n^{\frac{\alpha}{2}+\frac{1}{4}}\right) \int_{\frac{c}{n}}^{w} |\phi(x)| x^{\frac{\alpha}{3}+\frac{3}{4}} dx$$
$$= o\left(n^{\frac{\alpha}{2}+\frac{1}{4}}\right) \int_{\frac{c}{n}}^{w} \frac{|\phi(x)|}{x^{(\alpha+1)}} \left(x^{\frac{\alpha}{2}+\frac{1}{4}}\right) dx$$
$$= \left(n^{\frac{\alpha}{2}+\frac{1}{4}}\right) o\left(n^{\frac{-\alpha}{2}-\frac{1}{4}}\right) \int_{\frac{c}{n}}^{w} \frac{|\phi(x)|}{x^{\alpha+1}} dx$$
$$= \text{o}(1)\ o[\chi(n)]$$
$$= 0[\chi(n)], \text{ as } n \to \infty. \tag{2.11}$$

Considering I_3 and using (2.2), (2.3) and (2.5), we have

$$|I_3| = \int_w^n |\phi(x)| \left|L_n^{(\alpha+1)}(x)\right| dx$$
$$= \int_w^n |\phi(x)| e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} \phi\left(n^{\frac{2\alpha+1}{4}}\right) dx$$
$$= \text{o}\left(n^{\frac{2\alpha+1}{4}}\right) \int_w^n e^{\frac{x}{2}} x^{\frac{-(2\alpha+1)}{4}} |\phi(x)|dx$$

$$= o\left(n^{\frac{2\alpha+1}{4}}\right) \mathrm{o}\left(n^{\frac{-(2\alpha+1)}{4}} \chi(n)\right)$$

$$= o[\chi(n)], \text{ as } n \to \infty \qquad (2.12)$$

Again, considering I_4, using (2.2), (2.6) and (2.7), we have

$$|I_4| = \int_n^{\infty} |\phi(x)| e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} o\left(n^{\frac{6\alpha+5}{12}}\right) dx$$

$$= o\left(n^{\frac{6\alpha+5}{12}}\right) \int_n^{\infty} e^{\frac{x}{2}} x^{\frac{-(2\alpha+3)}{4}} |\phi(x)| dx$$

$$= o\left(n^{\frac{6\alpha+5}{12}}\right) \int_n^{\infty} \left(e^{\frac{x}{2}} x^{\frac{-(6\alpha+7)}{12}} x^{\frac{-1}{6}} |\phi(x)| dx\right)$$

$$= o\left(n^{\frac{6\alpha+5}{12}}\right) \mathrm{o}\left(n^{\frac{-1}{6}}\right) \int_n^{\infty} e^{\frac{x}{2}} x^{\frac{-(6\alpha+7)}{12}} |\phi(x)| dx$$

$$= o\left(n^{\frac{2\alpha+1}{4}}\right) o\left(n^{\frac{-(2\alpha+1)}{4}} \chi(n)\right)$$

$$= o[\chi(n)], \text{ as } n \to \infty \qquad (2.13)$$

Now combining (2.10), (2.11), (2.12) and (2.13), and using $f(0) = 0$, we get

$$|S_n(0) - 0| = o[\chi(n)], \text{ as } n \to \infty \text{ and therefore} |S_n| = o[\chi(n)], \quad \text{as } n \to \infty.$$

Therefore applying (2.7), we have

$$\sum_n \frac{|S_n|}{P_n} = \sum_n \frac{|\chi(n)|}{P_n} < \infty \quad \text{i.e. convergent.}$$

This completes the proof of the theorem.

Conclusion

In this paper, we have proved a theorem related to the degree of approximation of function belonging to L[0,∞) by Nörlund Summability of Fourier-Laguerre series at the end point $x = 0$. This work establishes some of the results that characterize the approximation relations of the function in $L[0,\infty)$ by Nörlund Summability of Fourier-Laguerre series. In fact, these results can be used for further study in many practical problems in science and engineering.

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