What are possible Sources of Common and Persistent Student Errors in Algebra and Calculus?

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Abstract

Research shows that college students make numerous algebra and other prerequisite content-related errors in Calculus courses. Most of these errors are common, persistent, and often observed in simple mathematical tasks. This qualitative study is an attempt to identify the potential sources of such errors. Based on our observations of student errors, we wrote a Precalculus and a Calculus test and administered them in twelve sections of four different undergraduate mathematics courses for which either Precalculus, Calculus I or both were a prerequisite. The tests were announced on the first day of the class and administered the following week. All the questions on the test were True or False questions. Based on our experience as college mathematics instructors, we assumed that many students would perceive the True answers as False and the False as True. Therefore, if students' selected a given answer, mathematical statement, process or solution as True, they were asked to justify why that was not False and vice-versa. They were instructed to provide logical explanations and avoid plugging in numbers to check for correctness. Analysis of data using grounded theory approach resulted in the following three possible external sources of common and persistent student errors: a) Difficulty with symbols and/or lack of attendance to the meaning of those symbols, b) Instructional practices, and c) Lack of knowledge. We will provide examples to illustrate how such errors could have originated from these sources.

Keywords: *Algebra, Calculus, Common Errors, Mathematical Meaning, Symbols*

Introduction

There is well-documented evidence that many students who take college mathematics courses often come underprepared (Poladian & Nicholas, 2013; Rodgers, Blunt, & Trible, 2014). Lack of prerequisite knowledge and skills are found to hinder student success in these courses (Booth et al., 2014; Stewart et al., 2018). From our own experience as college mathematics instructors, we observe that many students make errors even on relatively simple tasks. While making mistakes in mathematics is natural (Sousa, 2008), many of the mistakes are repetitive and common in nature (Elbrink 2007; Stavrou 2014, Benander et al. 1985 & Ashlock 2006). Students often make similar and predictable errors. For example, many students think $\sqrt{a^2+b^2}=a+b$ is a true statement even for nonzero numbers a and b. Despite continued effort, repeated observations of these errors can be frustrating to mathematics instructors. According to Stewart et al. (2018), many students also express frustration with their lack of algebraic skills, which they believe are inhibiting their success in solving calculus problems. However, they found that the same group of students were also made numerous errors on relatively simple calculus tasks that did not require algebraic and other prerequisite skills. This means that such repetitive common errors are not just limited to basic algebra.

Research shows that the error patterns displayed by students are not due to carelessness alone or lack of insufficient practice (Ashlock, 2006). Then a natural question to ask in this regard is: why do students make such common and repetitive errors? Finding a specific and definitive answer to this question is difficult and impractical because error types vary from problem to problem, and students make such errors for a number of reasons. Even though some research has identified and categorized student common algebraic errors in calculus (Ashlock, 2010), research on identifying possible sources of such errors is sparse. This qualitative study is an attempt to list the possible external sources that can be attributed to students' repeating common errors in algebra and calculus.

Literature Review

There can be three different types of error in mathematics: factual, procedural, or conceptual, and those errors may occur for a number of reasons (Lai, 2012). Students make errors primarily due to two reasons: lack of knowledge (Hudson & Miller, 2006), and carelessness and poor attention (Stein, Silbert, & Carnine, 1997). However, research also shows these are not the only reasons of error patterns displayed by students (Ashlock, 2006). This indicates that there are other sources of student errors that need to be identified.

The use of symbols that are used to express mathematical expressions, functions, and statements could be a source of student difficulties because they add a layer of abstraction to novice learners. According to Drouhard and Treppo (2004), although the language aspect of mathematics (that is, use of symbols) makes it possible to "move fluently through a layer of abstraction and compress complex mathematical thoughts into efficient symbol strings" (p. 240), such characteristics also make symbolic writings very "opaque" to novice learners. As a result, many students carry out procedures to 'solve' problems by manipulating the symbols but without attending to or understanding the meaning of those symbols (Harel, Fuller, & Rabin, 2008). Gray and Tall (1994) introduced the notion of “procept” to describe a combination of symbols that can be interpreted as a process, or as denoting a mathematical object of that process. In any given mathematical context, they consider students as fluent in algebra (and other areas of mathematics) if they have no difficulty in moving back and forth between these two viewpoints. Sfard and Linchevski (1994) see this duality of process and object as two different layers of mathematical understanding, which involves operational and structural modes of thinking. They introduced a notion of “reification” to explain these hierarchical levels of cognitive abstraction: moving from process to seeing it as an object in its own right. Since it involves a significant cognitive restructuring to move back and forth between these two levels of mathematical understanding, many students often face difficulties with symbolic representations of mathematics. According to them, how individuals interpret a collection of symbols depends on their preparedness to notice and ability to perceive.

Researchers suggested that students’ difficulty with symbols and their poor attendance to the meanings of those can be linked to certain teaching practices (Brousseau, 1997; Harel et al., 2008; Schoenfeld, 1991). Brousseau (1997) referred to this phenomenon as *didactical obstacles*. Teaching practices that deemphasize the meanings of symbols and definitions “may encourage the undesirable *non-referential symbolic way of thinking*, wherein students view symbols as having a life of their own and manipulate them based on arbitrary rules” (Harel et al., 2008, p. 125). Schoenfeld (1994) suggested that students’ overall impression of mathematics comes directly from the practices of their mathematics classrooms. Based on class observation of teachers, Harel et al (2008) suggested teachers’ use of “shortcut” rules and treatment of symbols and mathematical expression of certain forms as inputs to a standard procedure is deemphasizing the meaning of the symbols and expressions. According to them, such teaching practices can develop a belief in students that carrying out procedures is more important than understanding the meaning and reasoning. For this reason, if teachers do not attend to the meanings of symbols as they teach, students may develop a non-referential symbolic way of thinking (Harel, Fuller, & Rabin, 2008).

Impact of algebraic errors on students’ success in calculus courses has been well documented in the literature. Most of those errors are widespread among students, persistent, and are made on relatively simple tasks (e.g. Booth et al., 2014; Stewart et al., 2018). Students are also found to make errors on simple calculus tasks that do not require algebraic and other prerequisite skills (Stewart et al., 2018). Existing research is mostly limited to identifying and cataloguing algebraic and computational errors (e.g. Ashlock, 2010), but studies on identifying potential sources of common but persistent student errors is sparse. Using previously identified student difficulties in the literature and our own observation of student errors as college mathematics instructors, this research attempts to close this gap by identifying the possible sources of such errors. In particular, we aim to answer the research question: Where do students' common and persistent errors in algebra and calculus originate from?

Method

In the summer 2018, the authors met in a research workshop in which classroom experiences were shared among authors, and a decision to collaborate on this project was made. Two co-authors were teaching at small two-year colleges in the Midwest, one in a small private four-year college, another one in a medium-sized public university, and the other in a medium-sized private university in the southeastern United States.

Based on the review of literature and our own experience as college mathematics instructors, we developed a framework to design and guide our research as well as to understand the results of the study. Based on review of literature, we found that use of symbols in mathematics generally add a layer of abstraction making the learning process more difficult for the learners. Learning of any subject is a function of how students are taught, and what the students bring with them from their prior experience (e.g. Ball, 1988). Harel, Fuller and Rabin (2008) found that certain instructional practices and classroom culture hinder student learning (Harel, Fuller & Rabin, 2008). They found that mathematics teachers also provide "rules" to memorize, "keywords" to look for, and procedures (in addition to the concepts in most cases) to "solve" problems. In many cases, teachers are likely to teach the way they had experienced teaching as students, and students are also likely to follow "steps" or prescribed procedures to "solve" problems without even understanding or knowing what and why they are doing. Students carry out procedures to solve problems without understanding or attending to the meanings of the symbols being used. Such "teaching" and "doing mathematics" have been established as cultural phenomena among many teachers and students. Based on our own experience as mathematics instructors, we find that students do not always store conceptual knowledge, correct "rules", "kew words" and procedural steps in their brain, and in many cases, they overlook the contexts (or even blindsided by the instructional practices) in which those rules or procedures could be applied. Even if they have stored complete information, many of them can retrieve only partial information when they solve problems. When students have perceived any concepts or procedures as correct (even if they are not), research shows that "unlearning" is more difficult than learning (e.g. Lubelfeld & Polyak, 2017) and therefore, relearning to correct such misconceptions is difficult causing the errors to persist. In many cases, students learn and internalize correct concepts, and even retrieve them when they need to use them. However, they commit careless mistakes due to poor attention. We believe different combinations of these external factors eventually give rise to student errors, and so we have used this as our framework to design our research and understand the sources of student errors (see Figure 1). We want our readers to know that we are only seeking to identify possible external factors or sources of student errors. We understand that analyzing student thinking from the perspective of cognitive psychology is needed to determine the root cause of student learning.

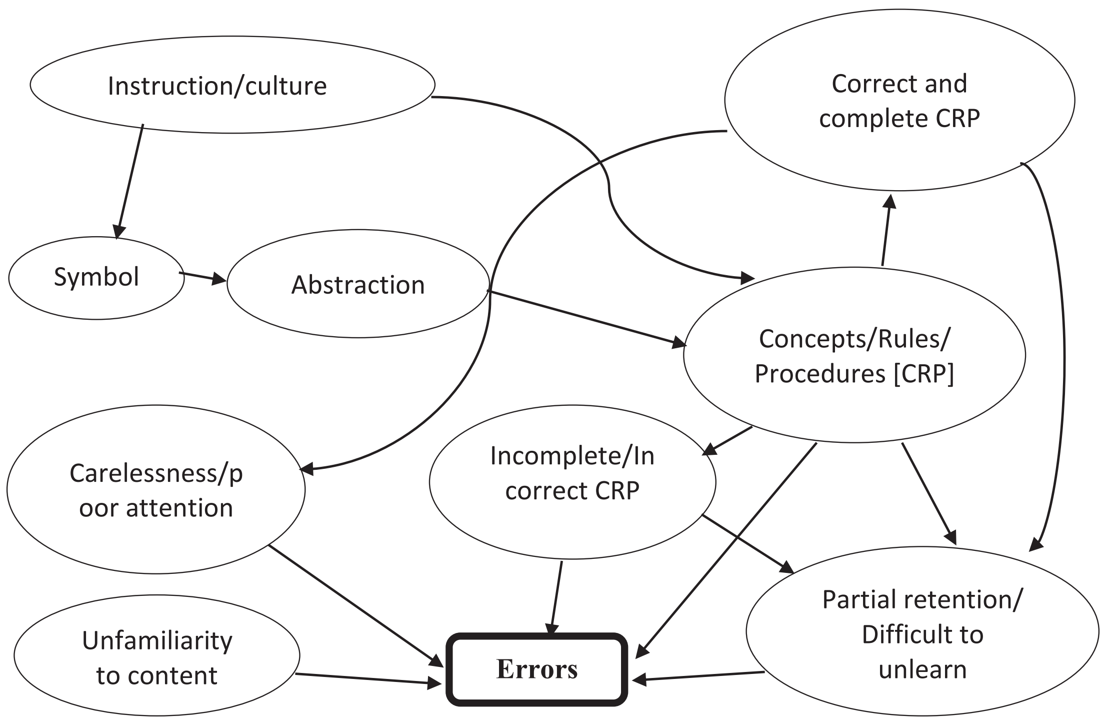

Figure 1. Our framework to understand external sources of student errors.

Based on this framework, and our past observation of student errors, we carefully developed and administered two tests (precalculus and calculus) in the Fall 2018 semester to assess students' prerequisite concepts and skills. We wrote problems that many students were assumed to make errors in their solutions (both tests can be found in the appendix), stemming from each of the sources as depicted in Figure 1. For example, some problems were written anticipating students' careless errors while some others were written anticipating errors due to abstraction brought by the use of symbols. The precalculus test included problems involving algebra, exponential, logarithmic and trigonometric functions that are taught in a typical precalculus course in the United States. Similarly, the calculus test included only basic limits, derivative and integral problems that are taught in a typical calculus I course. Each test included 12 problems and the students were given a 40 minutes time limit. The problems in the tests were True or False questions. We were interested in learning why students would perceive incorrect answers, statements or justifications as correct, and correct as incorrect. For this reason, if they selected True, we asked them to justify why that was not False. Similarly, if they selected False, they were asked to justify why that was not True. For example, if students choose the statement $\sqrt{a^2+b^2}=a+b$ as True, they were asked to justify why it was not False. We also asked students to avoid plugging in numbers to check for correctness and giving reasons such as "it is not True because it is False". Rather, they were asked to provide logical explanations in their justifications.

We selected our research participants using convenience sampling: those students who were enrolled in our own classes. After getting an IRB approval from the Institutional Review Board, we secured informed consent from the participants to voluntarily participate in the research. Since all the students were 18 years or older, and they all agreed to participate, no one was excluded from participating in the research. After the data was collected, student papers were scanned, and stored in password protected personal office computers removing all identifiable information of the students

The Precalculus test was administered in 12 sections of four undergraduate courses: Business Calculus, Calculus I, Calculus II, and Introduction to Differential Equations. The Calculus test was given to three sections of Calculus II, and one section of Introduction to Differential Equations. We announced the tests on the first day of the class and administered them the following week. We employed the constant comparison method (Strauss & Corbin, 1994), a Grounded Theory approach to analyze our qualitative data (student justifications). During our initial discussion, we decided to grade student responses in each problem and code supporting justifications to categorize them into one of the two predetermined categories: *right choice supported by right justifications (RR)* and *wrong choice supported by wrong justifications (WW)*. A random sample of 20 exams (a total of 240 responses) was selected which was coded independently by each author. Some students selected right answers but they provided either incorrect or unsatisfactory justifications or no justification at all. In our next meeting, we decided to categorize such responses into the third category: *right choice supported by unsatisfactory justifications (RU)*. Afterwards, we analyzed data collected from our own respective classes and categorized them into the three categories *RR*, *WW*, and *RU*.

In the next step, we focused our attention on the category *WW* because we were interested in understanding the sources of student errors in their justifications. We independently coded student justifications in each problem of the same random sample for possible sources of errors. We met again to discuss our codes, renamed our codes as necessary, and sorted out any discrepancies. The discussions continued until we reached 100% agreement in each problem. As we saw similar (but incorrect) justifications in multiple students' responses such as "this was a rule given in high school", we realized that we should have included student interviews in our research design. Since class observation was not an option (we were assessing students' prerequisite knowledge at the beginning of the semester), we also decided to use our experiences, perceptions, and reflections of our own teaching (such as how we had or would have taught the concepts/skills we were assessing) to help us in the coding process. Each of us then coded the data from our respective classes during which we sought each others' opinions multiple times if there were difficulties in coding any student responses. After multiple rounds of discussions, reading and rereading of student justifications, and sorting our codes, four refined categories of the sources of student errors emerged. The categories along with some illustrative examples will be discussed in the Results and Discussion section.

Results and Discussion

Altogether 232 students participated in the study. When the data from both tests were combined together, we found that 48.6% responses fell into the category RR, 12.8% into *RU*, and the remaining 38.6% into *WW*. Tables 1 and 2 provide the percentages of these categories in each problem. Note that the problems in the tests did not appear in the given order; they are ordered according to the highest to lowest WW rates.

Table 1

Percentage of categories WW, RU and RR by problems in the precalculus test

S.N.	Question	WW	RU	RR
1	The ONLY solution to the equation $x^2 = 4x$ is $x = 4$.	67%	10%	23%
2	$\frac{\log x}{\log y} = \log x - \log y$	63%	6%	31%
3	$e^{-3\ln t} = -3t$	60%	9%	31%
4	$\sqrt{x}(1+x^2) = \sqrt{x} + x$	38%	12%	50%
5	$x + \cos x = (1 + \cos)\, x$	36%	10%	54%
6	$\sqrt{a^2 + b^2} = a + b$	33%	18%	49%
7	$\sin 2x = 2\sin x$	32%	22%	46%
8	$\cos^{-1} x = \frac{1}{\cos x}$	27%	4%	69%
9	$\log x = 2$, then $x = 10^2$.	22%	16%	62%
10	The equation $x(x-2) = 4$ can be solved as $x = 4,\quad x - 2 = 4$ $x = 4,\quad x = 6.$	19%	31%	50%
11	If b the hypotenuse, and c, and a are the two legs of a right triangle, then $b^2 = c^2 + a^2$.	10%	5%	85%
12	$\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y},\ x \neq 0, y \neq 0$	9%	1%	90%

Table 2

Percentage of categories WW, RU and RR by problems in the calculus test

S.N.	Question	WW	RU	RR
1	Let a is a constant and x is a variable. Then $\frac{d}{dx}a^x = xa^{x-1}$.	44%	8%	48%
2	If x is a function of y, then $\frac{d}{dy}x^2 = 2x$.	38%	11%	51%
3	$\tan^{-1} x = \frac{1}{1+x^2}$	38%	4%	58%
4	$\frac{d}{dx}e^2 = 2e$	34%	11%	55%
5	$\frac{d}{dx}\sin(\cos x) = \cos(\cos x) + \sin(-\sin x)$	34%	10%	56%
6	$\int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \ln\|x\| + \ln\|x^2\| + C$	34%	8%	58%
7	$\frac{\sin x}{x} = 1$	33%	20%	47%
8	$\frac{d}{dx}x^{e^2} = e^2\, x^{e^2-1}$	33%	4%	63%
9	$\int \frac{\frac{1}{2}x^2 + x}{x^3} dx = \frac{1}{2}\int \frac{x^2+x}{x^3} dx$	33%	3%	64%
10	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$	23%	9%	68%
11	$\lim_{x\to 5} \frac{x-5}{x^2-25} = \frac{0}{0} = 0$	21%	15%	64%
12	$\int \ln x\, dx = \frac{1}{x} + C$	20%	14%	66%

Our analysis resulted in the following thee refined categories for possible sources of student errors. Codes, groups of codes, and final refined categories can be found in the Appendix B. We want our readers to know that these sources or factors should not be considered as complete, definitive, or mutually exclusive. We understand that it is not reasonable to attribute student errors directly to a particular source because most of the errors were carried over from the past (we were assessing students' prerequisite concepts and skills).

a) Difficulty with symbols and/or lack of attendance to the meaning of those symbols.

We found that students had either difficulty with the use of symbols or they did not attend to the meanings of those symbols. For example, a little more than one-third of the students considered the following derivative $\frac{d}{dx}\sin(\cos x) = \cos(\cos x) + \sin(-\sin x)$ as correct. In their justifications, they wrote that the product rule was correctly applied and that the derivatives of both Sine and Cosine functions were correct. This example shows that many students have difficulty with the use of symbols in denoting trigonometric functions and their compositions. For them, the function $\sin(\cos x)$ and the product $\sin x \cos x$ were the same. Despite treating $\sin(\cos x)$ as the product, they were probably thinking that it was okay to write the argument *x* in one of them and not in the other. For example, a student wrote that "the derivative of the product is $f(x).g(x) = f'(x)g(x) + f(x)\ g'(x)$" without writing the derivative symbol on the left side (see Figure 2).

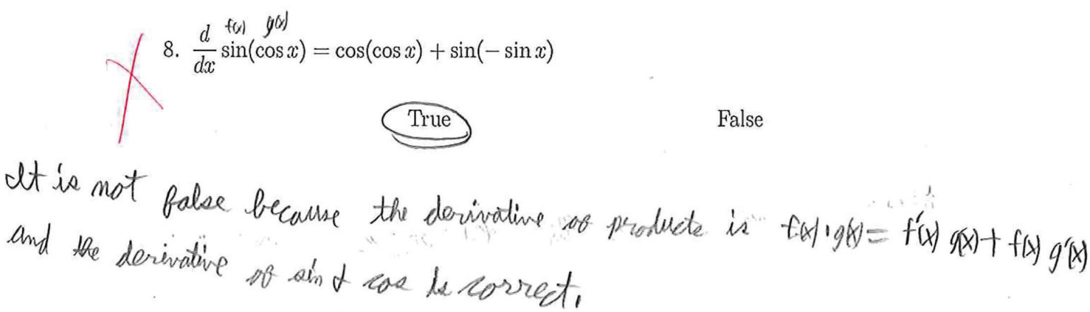

Figure 2. An example illustrating how students overlook symbolic representation of a composite function.

The student failed to see the distinction between the composition and the product of functions in the first place. Even though this particular student correctly remembered the product rule for derivatives, student's use of symbols to express the product rule is incorrect despite writing "the derivative of the product". The statement $f(x).g(x) = f'(x)g(x) + f(x)\ g'(x)$ would make no sense. This example shows students' difficulty with the use of symbols or their non-attendance to the meaning of the symbols they use to express mathematical statements.

We also observed that students often get confused with the same symbol to represent two different things. For example, we noticed that students often have difficulty in distinguishing the difference between the reciprocal of a function and its inverse primarily because of the use of the same superscript -1. A little more than a quarter of the students (27%) thought that the inverse of cosine function $\cos^{-1}x$ is the same as the reciprocal $\frac{1}{\cos x}$. In their justifications, they stated that since $x^{-1} = \frac{1}{x}$ is true, the above statement should be true as well for the same reason. Their confusion seems to have stemmed from our conventional use of the same superscript symbol -1 to denote the reciprocal of a quantity and the inverse of a function. Since students learn the reciprocal of numbers and the symbol to denote them earlier than the inverse of functions, it looks like they find it difficult to use the same symbol for the inverse functions.

Around 38% of all the students marked the statement $\tan^{-1}x = \frac{1}{1+x^2}$ as True, which came as highly surprising to us. One of the students wrote "This is not False because the basic derivative rule of arctan gives this". It is worth mentioning that the statement did not have a derivative or integral symbol. However, this student assumed that the derivative symbol was there, but it was either not necessary or not important in the statement. We considered justifications like this as examples of students' lack of attention to the meaning of the symbols. This particular student justification also shows how students incorrectly verbalize the mathematical statements expressed with the use of symbols. Similarly, some students justified the statements $\tan^{-1}x = \frac{1}{1+x^2}$ and $\frac{\text{Sinx}}{x} = 1$ as true by writing these are "the standard trig identities we learned in high school (or college)". As we know, there needs to be either a derivative symbol on the left or an integral symbol on the right for the first statement to be True and a limit (as x approaching zero) symbol for the second statement. However, students either did not pay attention to those symbols and their meanings or they were not able to recall the operations such as derivative and integral operating on one of those functions, or they just thought they were not important.

In many cases, several symbols are used to express a mathematical statement. Even though such denotations and statements are trivial for experts, they could actually be very difficult for novice learners to comprehend. For example, around 33% of the students marked the statement $\frac{\text{Sinx}}{x} = 1$ as True. They provided justifications such as "it is a known trig identity" and "this is a rule we learned in calculus". Many students perhaps know that the Sine function oscillates but the function $f(x) = x$ does not. Most likely, they can even correctly recall the graphs of these functions. However, the students seem to $\frac{\sin x}{x}$ as a constant function whose value is 1 for all x. Therefore, despite having the correct information at hand, many students did not pay attention to the "ratio" of those two varying quantities. Actually, we wanted to check if the students pay attention to the meaning of the limit symbol in the statement $\lim_{x\to 0} \frac{\sin x}{x} = 1$. This is a function whose limiting value as *x* approaches zero is discussed using numerical, graphical, geometrical, and even using L'Hopital's Rule in a standard Calculus I course in the US. In this statement, there are many symbols involved: the limit, the variable x, the function $\frac{\sin x}{x}$, the equal sign, and the numbers 0 and 1. Among these pieces of information, students perhaps stored only partial information, or even if they had stored the complete information, they retrieved those that stood out clearly to them: only $\frac{\text{Sinx}}{x}=1$ in this case. Based on student justifications, it is likely that students would consider the statement $\lim_{x\to 0}\frac{\text{sinx}}{x}=1$ as True if the number zero was replaced by infinity or any other number.

b) Instructional practices (use of memorized "rules", keywords or forms of expression).

Students were found to use many memorized "rules" that could have been either given by their instructors or made by themselves. In many cases, their rules consisted of only partial information. Students either memorized partial information given by their instructors, or they were not able to retrieve the complete information. Some important conditions under which their "rules" would be true were missing in their rules. For example, 60% of the students marked the statement $e^{(-3\ln t)}=-3t$ as True, and based on their justification, almost all of them selected their answer based on the same rule: "e and ln cancel each other out". In their written justifications, students implied that they could "cancel e and ln out" as long as they see both of them in the same expression. Similarly, 63% of all the students marked the statement $\frac{\text{logx}}{\text{logy}}=\text{logx}-\text{logy}$ as True providing the same justification: "product gives sum, division gives difference" (see Figure 3).

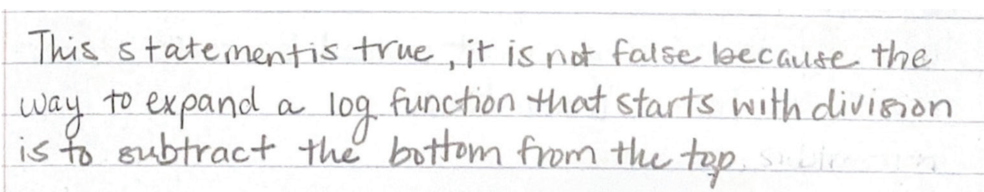
This statementis true, it is not false because the way to expand a log function that starts with division is to subtract the bottom from the top.

Figure 3. An example of a student's use of memorized rule.

Students perhaps stored and retrieved the key words 'logarithmic function', 'division or quotient', and 'difference' from the above "rule". In the same manner, one-third of all the students considered the statement $\sqrt{a^2+b^2}=a+b$ as True because "square and square root cancel each other out", a "rule" that they made by themselves or given by their teachers or someone else. Student justifications indicated that students are often misled by such rules because they seem to be caring less "under what conditions" their "rules" could be applied. Interestingly, 38% of all the students answered that $\sqrt{x}$ was correctly distributed in $\sqrt{x}\left(1+x^2\right)=\sqrt{x}+x$. They believed that this statement should be True because "square and square root cancel each other out". This indicates that they tend to "cancel" the square and square root as soon as they see those in a given expression without attending to the underlying operations.

Many students are found to have paid more attention to the "forms" of the expressions to follow memorized procedures. For example, some students thought that it was okay to solve the equation $x(x-2)=4$ by equating each factor on the left side to the number on the other side (see Figure 4). In this case, students treated

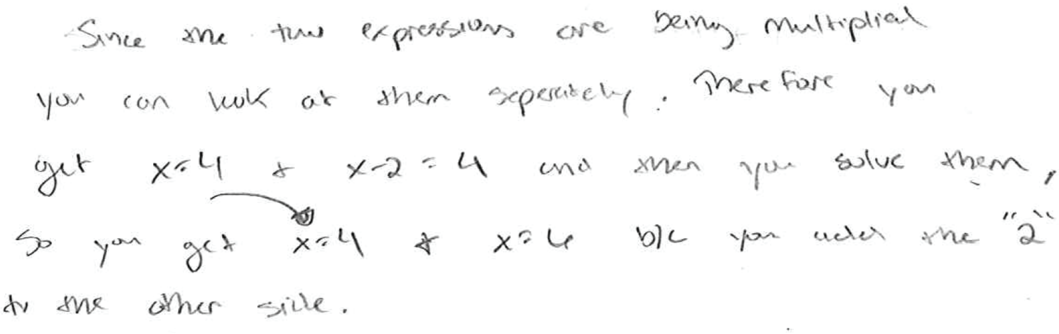

Figure 4. An example showing how students use forms of expressions as a guide to use a procedure without attending to the meaning of the expression.

Similarly, 34% of the students answered the statement $\int 1/x\,dx + \int 1/x^2 dx = \ln x \quad \ln x \quad C$ $\ln|x| + \ln|x^2| + C$ as correct. In their justifications, they indicated that since $\int 1/x\,dx = \ln|x| + C$ is true, the same should be true for $\int 1/x^2 dx = \ln|x^2| + C$. Interestingly, 20% of all the students also answered the statement $\int \ln x\,dx = 1/x + C$ as True in the same test. In the later case, students' memorized rule was "lnx gives 1/x, and 1/x gives lnx." They paid attention to only the functions but not much to the operators (differential or integral) or could not retrieve the correct operations on the correct functions.

Class observation was not a part of our data collection because we were assessing students' prerequisite concepts and skills at the beginning of the semester. However, after reading student justifications and existing literature, we reflected on our own teaching practices and agreed that many of the student errors could have come from our own classroom practices. We understand that we need class observations and more data to back up any claims. Therefore, we only speak for ourselves here. There are several concepts, skills or symbols that are trivial for us (instructors) but confusing and difficult for students to comprehend. When we teach, we often overlook (unintentionally) their difficulties and how they process information in their minds, and internalize concepts and procedures, which could have unintended consequences in student learning. For example, while introducing the concept of and symbols to denote inverse functions, we could be overlooking the fact that the use of the same superscript -1 to denote reciprocals and inverse functions could be confusing students. A brief review of the reciprocal of numbers or functions and their symbols before introducing the concept of inverse functions and their symbol could have helped. Similarly, after introducing the concept of inverse of a function, a simple example such as "the reciprocal of a nonzero number x is $x^{(-1)} = 1/x$ but the inverse of a function $f(x) = x$ is itself, that is $f^{-1}(x) = x$" could help alleviate student confusion between the reciprocals and inverses. Similarly, explaining conventional use of the superscript -1 in $\sin^{-1}x$ is reserved only to denote the inverse sine function but the reciprocal of sinx is written as (which is cscx) could help. Comparing the graphs of reciprocal and inverse functions could help the students further to see the distinction between the two. When we reflected on our teaching practices, we realized that we just introduced the concept of inverse of a function and the notation used to denote inverse functions without stressing much on the distinction of the two concepts despite the similarity of the symbols used.

When we reflected on our teaching practices, we all agreed that we showed (or had students show) the standard limit example $\lim_{x\to 0} \frac{\sin x}{x} = 1$ using numerical, graphical, geometrical, and later even using the L'Hopital's Rule. Still, several students marked the statement $\frac{\sin x}{x} = 1$ as True. In their justifications, students wrote that "it is a known trig identity" or "this is a rule we were given". In the limit statement $\lim_{x\to 0} \frac{\sin x}{x} = 1$ there are many symbols involved: the limit, the variable x, the function $\frac{\sin x}{x}$, the equal sign, and the numbers 0 and 1. Based on student justifications, we can see that some of the information either goes unnoticed or is hidden from their view or students simply think it is okay to not write all of the symbols. As a result, the students perhaps store only partial information, or even if they have stored the complete information, they are able to retrieve only those that stand out clearly to them: only $\frac{\sin x}{x} = 1$ in this case. When we reflected on our teaching, we realized that we did not either emphasize enough on the meaning of the limit symbol or the covariation between the variable and the function, or did not stress the importance of the domain or the graph of the function. As a result, the students either did not understand the meaning of the limit symbol and its importance in the statement $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the number where *x* was approaching (zero in this case), and therefore, the limit symbol could have become less important or even non-important to them. We showed the graph of the function $\frac{\sin x}{x}$ (or had students graph it) but did not ask them to find the limit as x approached other numbers rather than just approaching zero (even if the function is continuous elsewhere). If we (and other instructors) do this, perhaps the students would see the covariation of the independent variable and the function, and that the limit would not always be 1.

Similarly, as we reflected on our teaching, we explicitly (or would have) mentioned that "if the product of two real numbers is zero, either one or both of them must be zero" while teaching how to solve equations like $x(x-2) = 0$. We assumed that this should be a simple concept and students should have no difficulty internalizing it. However, after analyzing student justifications, we observed that they pay more attention to the form of the expression and the process rather than to what role the number zero on the right-hand side of the equation plays (see Fig 3 above). As a result, when students saw equations of similar form $x(x-2)=4$, they tend to separate x=4 and x–2=4 to solve. As we discussed how we would have taught to solve this equation, we all agreed that we would not stress enough on the role the number zero on the right-hand side plays (we would have certainly mentioned it once) and perhaps would not even give an example to explain why the process would be different if the number on the right-hand side was not zero as in $x(x-2)=4$. This made us realize that our instructional practice could be a cause of student errors. Similarly, if we were to teach inverse of functions, at one point, we would have come to explain why $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$ in their domains but we could have easily forgotten to give examples to explain why $f\left(cf^{-1}(x)\right)$ is not always equal to cx for a constant c. Our action can easily cause students to make their own rule such as "f and f^{-1} cancel each other out" and "e and $\ln$ cancel each other out". As a result, student error in considering the statement $e^{-3\ln t} = -3t$ as True can be attributed (at least partially) to our classroom practices. About 60% of the students considered this statement as True in our data.

c) Lack of knowledge.

This factor or source is the least interesting for us as researchers, perhaps an easy guess to the readers, and also documented in the literature (Hudson & Miller, 2006). In many cases, students simply did not have the knowledge of some of the mathematical facts or conventions, and as expected they were likely to make errors. In most such cases, it was not easy for students to check the correctness of the 'facts' they thought were true or not true because checking the correctness would also require knowledge. For example, 22% of the of all the students considered the statement "If $\log x = 2$ then $x = 10^2$" as False, and 44% of them considered $\frac{d}{dx}a^x = xa^{x-1}$ as a True statement (for a positive constant a). To be able to check the correctness of these statements, they needed to know the definition of the exponential and logarithmic functions, the distinction of the exponential and power functions, and how their derivative formulas were derived. Similarly, 23% of all students did not know the correct derivative of an inverse Sine function, which eventually made it difficult for them to check the correctness of the antiderivative.

The factors or sources discussed above were investigated individually for their impacts on student learning. However, we did not find any studies that attempted to link common and persistent errors (especially related to calculus concepts) to these factors. This study has been able to fill that gap in the literature to some extent. There are documented evidences of students' challenges with mathematical symbols and their non-attendance to the meaning of those symbols (e.g. Bigg & Pierce, 2020; Drouhard & Treppo, 2004; Harel, Rabin & Fuller, 2008; Rubenstein & Thompson, 2001). According to Rubenstein and Thompson (2001), mathematical writing with the use of symbols challenges our students, and as a result, they face difficulty in verbalizing, reading, understanding, and expressing their mathematical ideas in writing. While the use of symbols in mathematics enables us to express abstract and compress complex mathematical ideas efficiently in writing, these symbolic writings add another layer of abstraction to novice learners (Drouhard & Treppo, 2004). As a result, students cannot move back and forth easily between the process and object viewpoints of the combination of symbols used to denote a mathematical expression (Gray and Tall, 1994). Since it involves a significant cognitive restructuring to move back and forth between these two levels of mathematical understanding, many students often face difficulties with symbolic representations of mathematics (Sfard and Linchevski, 1994). Many students, therefore, just look at the forms of expressions and follow a set of procedures to 'solve' problems by manipulating the symbols but without fully understanding or attending to the meaning of those symbols (Harel, Fuller, & Rabin, 2008).

There can be other factors or sources of student misconceptions and some of these difficulties could be inherent to the subject itself. However, most of the potential sources of common and persistent student errors could be the result of instructional practices. Observing classes of high school mathematics teachers, Harel, Fuller, & Rabin (2008) noticed various issues concerning purpose, mathematical terminologies, and mathematical symbols. There were several instances where the instructors demonstrated no intellectual purpose for what they were teaching, mathematical symbols were either ignored in many cases, or were presented without giving their meanings. In some cases, the symbols were also treated as inputs to procedures. Similarly, more emphasis was given to the forms of expression, procedures were treated as more important than the meaning of the terms, and the term short-cut was used to show quicker computations shadowing the underlying concepts. Likewise, they noticed that either incorrect definitions were accepted in some cases or multiple definitions were accepted without even discussing their equivalences or differences. Summarizing the perceptions of mathematics faculty and tutors, Begg and Pierce (2021) state that the students transitioning from high school to college have difficulty with symbols not only because they are not familiar with the symbols and syntax but also because the symbols and syntax template they are already familiar with also take on a different meaning or range of meanings in college mathematics courses (Begg & Pierce, 2021). The faculty and tutors in that research also believe that the comprehension of symbols in newer context and extended domain should be explicitly taught to the students.

Implication for Instruction

We believe that our classroom practices should align with what we expect our students to learn. As the results suggested, mathematical symbols and instructional practices were two major sources of student errors, mathematics instructors' assessment instruments should value mathematical communication skills, and continuously reflect on their teaching practices. Moreover, they should also collect and analyze the cognitive aspect of common student errors to find the root cause of such errors. And since it can be difficult to unlearn what has been already "learned", middle and high school teachers should be made aware of common difficulties of students transitioning from schools to college, and be provided training and resources for early intervention. We end this paper by endorsing Schoenfeld's (1994) statement: "the activities in our classrooms can and must reflect and foster the understandings that we want students to develop with and about mathematics." (p. 60).

There are some limitations to this study. The institutions from which the data was collected were not randomly selected. The sources of student errors that we came up with were solely based on the analysis of written justifications provided by the students, and using our reflections, experience and perceptions. Future research should include classroom observations, interviews of students as well as those of school and college mathematics instructors, and the data should also be analyzed from the perspective of cognitive psychology to find out the root causes of common and persistent student errors.

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Appendix A

Precalculus Test

MATH*** Precalculus Exam ID:

Instruction: Circle True or False under each statement. If your answer is True, explain why it is Not False, and if your answer is False, explain, why it Not True. Do Not check your answer by plugging in number; provide logical explanations to each of your answer.

1. The ONLY solution to the equation $x^2 = 4x$ is $x = 4$

True False Justification

2. $\frac{logx}{logy} = logx - \log y$

True False Justification

3. $e^{-lnt} = -3t$

True False Justification

4. $\sqrt{x}\,(1 + x^2) = \sqrt{x} + x$

True False Justification

5. $x + cosx = (1 + \cos)\mathrm{x}$

True False Justification

6. $\sqrt{a^2 + b^2} = a + b$

True False Justification

7. $sin2x = 2sinx$

True False Justification

8. $cos^{-1}x = \frac{1}{cosx}$

True False Justification

9. $If \log x = 2, then\ x = 10^2$

True False Justification

10. A student solved the equation $x(x-2) = 4$ as follows, x = 4, x-2 = 4 and got x = 4 and x = 6 his answers. His solution is

True False, Justification

11. If b the hypotenuse and c and a are the two legs of a right triangle, then $b^2 = c^2 + a^2$

True False, Justification

12. $\frac{1}{x+Y} = \frac{1}{x} + \frac{1}{y}, x \neq 0; y \neq 0$

True False, Justification

Calculus Test

MATH*** Calculus Exam ID:

Instruction: Circle True or False under each statement. If your answer is True, explain why it is Not False, and if your answer is False, explain, why it Not True. Do Not check your answer by plugging in number; provide logical explanations to each of your answer.

1. Let a is a constant and x is a variable. Then $\frac{d}{dx}a^x = xa^{x-1}$

   False True Justification

2. If x is a function of x, then $\frac{d}{dx}x^2 = 2x$

   False True Justification

3. $tan^{-1}x = \frac{1}{1+x^2}$ False True Justification

4. $\frac{d}{dx}e^2 = 2e$ False True Justification

5. $\frac{d}{dx}\sin(cosx) = \cos(cosx) + \sin(-sinx)$

   False True Justification

6. $\int\left(\frac{1}{x} + \frac{1}{x^2}\right) = ln|x| + ln|x^2| + C$

   False True Justification

7. $\frac{sinx}{x} = 1$

   False True Justification

8. $\frac{d}{dx}x^{e^2} = e^2x^{e^2-1}$

   False True Justification

9. $\int\frac{\frac{1}{2}x^2+x}{x^3}dx = \frac{1}{2}\int\frac{x^2+x}{x^3}$

   False True Justification

10. $\int\frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + C$

    False True Justification

11. $\lim_{x\to5}\frac{x-5}{x^2-25} = \frac{0}{0} = 0$

    False True Justification

12. $\int lnx\, dx = \frac{1}{x} + C$

    False True Justification

Appendix B
Codes and Groups

Codes	Groups	Themes
Inverse functions as reciprocals Composition of functions as their product	Multiple use or meaning of same symbol/similar looking expressions	Difficulty with symbols/lack of attention to meanings of symbols
Understanding parts but not the whole Treating function names as quantities Illegal distribution (of operators, function names, exponents, symbols, fractions etc.) Illegal swapping of symbols or operators Difficulty with function and inputs Difficulty with relationship between variables Not using or inappropriate use of equal sign Assuming symbols/operators (limits, differential and integral) without being given Assuming symbols/operators without writing them to follow procedures	Abstraction/difficulty with meaning of symbols	
My teacher gave me this (rule, procedure, identity etc.) We were told to do this if … I know this because I learned it in high school We used to do this	Teacher	Instructional practices
Square and square root cancel each other out Natural log and e cancel each other out In log, multiplication gives sum and division means subtraction Can pull constants out (of operators) This is a rule This is an identity Using forms of expression as a way of selecting a procedure	Given or made up rules/ procedures	
No idea I don't know I never learned this I am seeing this for the first time	Unfamiliar to content	Lack of knowledge
My mind is blank right now Perhaps because …	Internalization/retrieval	