Common Fixed Point Theorems Using Compatible Mappings of Type (R)inSemi-metric Space

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Abstract

In this paper, we establish a common fixed point theorem for three pairs of self mappings in semi-metric space using compatible mappings of type (R) which improves and extends similar known results in the literature.

Keywords: Semi-metric space, compatible mapping of type R, common fixed point.

Introduction

The fixed point theory has become a part of non-linear functional analysis since 1960. It serves as an essential tool for various branches of mathematical analysis and its applications. Polish mathematician S. Banach published his contraction Principle in 1922. In 1928, K. Menger[7] introduced semi-metric space as a generalization of metric space. In 1976, M. Cicchese [4] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces.In 1986, G. Jungck [6] introduced the notion of compatible mappings. In 1994, Pathak, Chang and Cho [8] introduced the concept of a new type of compatible mappings called compatible mappings of type (P). In 2004, Rohen, and Ranjit [9] introduced the concept of compatible mapping of type of (R) by combining the definitions of compatible mappings and compatible mappings of type (P).

Let *X* be a non-empty set and $d: X \times X \to [0, \infty)$.Then, (X, d) is said to be a **semi-metric space** (symmetric space) if and only if it satisfies the following conditions:

W1:$d(x, y) = 0$ if and only if $x = y$, and

W2:$d(x, y) = d(y, x)$ for any $x, y \in X$.

The difference of a semi-metricspace and a metric space comes from the triangle inequality.

Definition 1.1 **[1]** Let *A* and *B* be two self-mappings of a semi-metric space (X, d). Then, *A* and *B* are said to be compatibleif $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 1.5[10]Let *A* and *B* be two self-mappings of a semi-metric space(X, d). For each $x \in X$ is said to be a **commuting point** of *A* and *B* if$ABx = BAx$.

Definition 1.6[9]Let*A* and *B* be two self-mappings of a semi-metric space(X, d). Then *A* and *B* are said to be **compatible mapping of type (P)** if for all $x \in X$ $\lim_{n\to\infty} d(AAx_n, BBx_n) = 0,$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, forsome$t \in X$.

Definition 1.7[9] Let *A* and *B* be two self-mappings of a semi-metric space(X, d). Then *A* and *B* are said to be **compatible mapping of type (R)** if for all
$x \in X$ $\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$and $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some$t \in X$.

Proposition 1.1 [10] Let A and B be two self-mappings of a semi-metric space(X, d). If a pair $\{A, B\}$ is compatible of type (R) on X, forsome$z \in X$, then we have

i) $\lim_{n\to\infty} AAx_n = Bzand\lim_{n\to\infty} ABx_n = Bz$ if B is continuous,

ii) $\lim_{n\to\infty} BBx_n = Azand\lim_{n\to\infty} BAx_n = Az$ if A is continuous and

iii) $ABz = BAz$and$Bz = Az$ if A and B are continuous at z.

In order to establish our result, we consider a function $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

($\emptyset$1)$0 < \emptyset(t) < t$, for $t > 0$, and ($\emptyset$2) for each $t > 0$, $\lim_{n\to\infty} \emptyset^n(t) = 0$.

Main Results

Theorem 2.1. Let(X, d)be a semi-metric space. Let A, B, T, S, P and Q becontinuous self- mappings of X such that

(i) $\{AB, P\}$and $\{TS, Q\}$are compatible mappings of type (R) and

(ii)$d(ABx, TSy) \le \emptyset(max\{d(Px, Qy), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)],$

$\frac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\}$ for all $(x, y) \in X \times X$,

then AB, TS,P and Q have a unique common fixed point. Furthermore, if the pairs (A, B) and (T, S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

Proof: Suppose $\{TS, Q\}$is compatible mappings of type (R), $\lim_{n\to\infty} TSTSx_n = \lim_{n\to\infty} Q\,Qx_nand\lim_{n\to\infty} TSQx_n = \lim_{n\to\infty} Q\,TSx_n$whenever $\{x_n\}$ is sequence in Xsuch that $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} Q\,x_n = v$, for some $v \in X$.This implies that $TSv = Qv = z$ (say).Since compatible mappings of type (R) implies commutative at coincidence point. So from proposition 1.1, we have $Qv = QTSv$. This implies $TSTSv = TSQv = QTSv = QQv$. So that for a given$v$, we have $TSz = Qz$whenever,$TSv = Qv = z$.

Similarly, if {AB, P} is compatible mapping of type (R), using proposition 1.1, we have $ABu = Pu = w$(say), since compatible mappings of type (R) implies commutativity at coincidence point. So, we have $ABPu = PABu$. This implies$ABABu = ABPu = PABu = PPu$. So that for a given u, we have $ABw = Pw$ whenever $ABu = Pu = w$.

We claim that $ABABu = TSz$. If not, then putting $x = ABu$ and $y = z$ in condition (ii), we get

$$d(ABABu, TSz) \le \emptyset(max\{d(PABu, Qz), \frac{1}{2}[d(ABABu, PABu) + d(TSz, Qz)],$$

$$\frac{1}{2}[d(ABABu, Qz) + d(TSz, PABu)]\})$$

$$= \emptyset(max\{d(ABABu, TSz), \frac{1}{2}[d(ABABu, ABPu) + d(Qz, Qz)],$$

$$\frac{1}{2}[d(ABABu, TSz) + d(TSz, ABABu)]\})$$

$$= \emptyset(max\{d(ABABu, TSz), 0, d(ABABu, TSz)\})\{\text{since } ABPu = ABABu\}$$

$$= \emptyset(d(ABABu, TSz))$$

$$< d(ABABu, TSz).$$

This is a contradiction. So, we get $ABABu = TSz$. Therefore, we have

$ABw = Pw = TSz = Qz$.

We claim that $ABu = TSz$. If not, then putting $x = u$ and $y = z$ in condition (ii), we get

$$d(ABu,TSz) \le \emptyset(max\{d(Pu,Qz), \frac{1}{2}[d(ABu,Pu) + d(TSz,Qz)], \frac{1}{2}[d(ABu,Qz) + d(TSz,Pu)]\})$$

$$= \emptyset(max\{d(ABu,Qz), \frac{1}{2}[d(Pu,Pu) + d(Qz,Qz)], \frac{1}{2}[d(ABu,TSz) + d(TSz,ABu)]\}$$

$$= \emptyset(max\{d(ABu,Qz), 0, d(ABu,TSz)\})$$

$$= \emptyset(d(ABu,TSz))$$

$$< d(ABu,TSz).$$

This is a contradiction. Therefore, we get $ABu = TSz$. Hence, we have

$ABu = TSz = Qz = Pu = ABw = ABPu = PABu = ABABu$. It follows that ABu is a common fixed point of AB and P.

We claim that $TSz = z$. If not, putting $x = u$ and $y = v$ in condition (ii), we get

$$d(TSz,z) = d(ABu,TSv)$$

$$\le \emptyset(max\{d(Pu,Qv), \frac{1}{2}[d(ABu,Pu) + d(TSv,Qv)], \frac{1}{2}[d(ABu,Qv) + d(TSv,Pu)]\})$$

$$= \emptyset(max\{d(ABu,Qv), \frac{1}{2}[d(Pu,Pu) + d(Qv,Qv)], \frac{1}{2}[d(ABu,TSv) + d(TSv,ABu)]\})$$

$$= \emptyset(max\{d(ABu,z), 0, d(ABu,z)\})$$

$$= \emptyset(d(ABu,z))$$

$$< d(TSz,z).$$

This is a contradiction. Therefore, we get $TSz = z$. Thus, we have

$z = ABu = TSz = Qz = Pu$. It follows that z is a common fixed point of TS and Q. Since $z = ABu$, z is common fixed point of AB, TS, P and Q.

For uniqueness, let z_0 be another common fixed point of AB, TS, P and Q. Then by putting $x = z$ and $y = z_0$ in condition (ii), we get

$$d(z,z_0) = d(ABz,TSz_0)$$

$$\le \emptyset(max\{d(Pz,Qz_0), \frac{1}{2}[d(ABz,Pz) + d(TSz_0,Qz_0)], \frac{1}{2}[d(ABz,Qz_0) + d(TSz_0,Pz)]\})$$

$= \emptyset(max\{d(z,z_0), \frac{1}{2}[d(z,z) + d(z_0,z_0)], \frac{1}{2}[d(z,z_0) + d(z_0,z_0)]\})$

$= \emptyset(max\{d(z,z_0), 0, d(z,z_0)\})$

$= \emptyset(d(z,z_0))$

$< d(z,z_0)$.

This is a contradiction. Therefore, we get $z = z_0$. Thus *AB*, *TS*, *P* and Q have unique common fixed point.

Finally, we need to show that z is only the common fixed point of mappings A, B, T, S, P and Q. Suppose the pairs (*A*, *B*) and (*T*, S) are commuting pair.

For this, we can write,

$Az = A(ABz) = A(BAz) = AB(Az)$. This implies that $Az = z$. Also,

$Bz = B(ABz) = BA(Bz) = AB(Bz)$. This implies that $Bz = z$.

Similarly, we get $Tz = z$ and $Sz = z$.

Hence *A*, *B*, *T*, *S*, *P* and Q have a unique common fixed point.

Example 2.1. Consider $X = [0,2]$ with the semi-metric space (X,d) defined by $d(x,y) = (x-y)^2$. Define continuousself mappings*A*, *B*, *T*, *S*, *P* and Q as $Ax = \frac{x+1}{2}$,$Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $S(x) = \frac{x+3}{4}$, $P(x) = \frac{3x+1}{4}$ and $Q(x) = \frac{2x+3}{5}$. Also define the sequence $x_n = \frac{1}{n} + 1$. Then, the mappings satisfy all the conditions of the above

Theorem 2.1 and hence we have a unique common fixed point at $x = 1.$

Now we have the following corollaries.

In the above Theorem 2.1, if we take $A = B$and $T = S$, then we have the following corollary.

Corollary 2.1 Let(X,d)be a semi-metric. Let *A*, *T*, *P* and *Q* be self- mappings of *X* such that

(i) $\{A,P\}$and $\{T,Q\}$are compatible mapping of type (R), and

(ii)*d*(*Ax*, *Ty*) $\leq \emptyset(max\{d(Px,Qy), \frac{1}{2}[d(Ax,Px) + d(Ty,Qy)], \frac{1}{2}[d(Ax,Ty) + d(Ty,Tx)]\})$for all $(x,y) \in X \times X$,then*A*, *T*,*P* and Q have a unique common fixed point.

In Theorem 2.1, if we take $A = B = Q$ and $T = S = P$, then we have the following corollary.

Corollary 2.2 Let(X, d)be a semi-metric. Let *A*and *T*be self- mappings of *X* such that

(i) A and $T$are compatible mapping of type (R), and

(ii) *d*(*Ax*, *Ty*) $\leq \emptyset(max\{d(Tx,Ay), \frac{1}{2}[d(Ax,Tx) + d(Ty,Ay)], \frac{1}{2}[d(Ax,Ty) + d(Ty,Tx)\})$for all $(x,y) \in X \times X$,

then *A* and *T* have a unique common fixed point.

Conclusion

Our result generalizes the result of U. Rajopadhyaya, K. Jha and Y. J. Cho [11], D. Sinha [10]and improves other similar results in the semi-metric space.

In 2014, U. Rajopadhyaya, K. Jha and Y. J. Chohave proved the fixed point theorem in semi-metric space using occasionally converse commuting however we have used the compatible mapping of type (R) to prove the same theorem. The theorem is the generalization of the result of U.Rajopadhyaya, K. Jha and Y. J. Cho as we have used the different tools to prove the same theorem.

In 2014, D. Sinha proved the fixed point theorem in metric space using compatible mapping of type (R) for four mappings, however we have proved the same theorem in semi-metric space using compatible mapping of type (R) for six mappings. The theorem is the generalization of the result of D. Sinha as we have used the different tools to prove the same theorem.

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