Dynamic Analysis of Prey-Predator Model with Harvesting

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Abstract

Employing the Lotka -Voltera (1926) prey-predator model equation, the system is presented with harvesting effort for both species prey and predator. We analyze the stability of the system of ordinary differential equation after calculating the Eigenvalues of the system. We include the harvesting term for both species in the model equation, and observe the dynamic analysis of prey-predator populations by including the harvesting efforts on the model equation. We also analyze the population dynamic of the system by varying the harvesting efforts on the system. The model equation are solved numerically by applying Runge - Kutta fourth order method.

Keywords: Harvesting efforts; Ordinary differential equation; Prey-predator model; Runge - Kutta fourth order method.

Introduction

Two species survive in the same region, and one as a predator which depends on the population of another as a prey. A common assumption in population studies is that, at the beginning, a population of one species will grow at a rate which is proportional to its initial size. The population changes as a function of time which describes this time-continuous phenomenon by means of an ordinary differential equation. If one species predates another affecting the population dynamics of both species, then the populations are in a prey-predator situation.

Malthus (1798) developed a continuous population growth model. He proved that the population is increasing exponentially if the birth rate is greater than the death rate and dies out if the birth rate is less than death rate. A predator-prey model equation in the form of a system of ordinary differential equation was derived by (Lotka, 1920; Voltera 1926). Chattopadhyay et al. (1999) studied the combined harvesting of a prey-predator system in which both the prey and predator would obey the law of logistic growth. They included the disease factor term in the model equation, and studied the dynamical behavior of the system. Xiao & Jennings (2005) studied the dynamical properties of the ratio-dependent predator-prey model with nonzero constant rate harvesting. Das (2016) modified the system of ordinary differentiable equation (ODE) of two variables into the system of ODE with four variables introducing the variables as susceptible prey, infected prey, susceptible predator, and infected predator. He observed the role of harvesting on the overall dynamics of the proposed system. Kunwar (2019) observed in the phase plot, and analyzed the prey-predator model. Pokhrel (2019) analyzed the Lotka- Voltera model equation by doing the variation of the interaction rate of the two species.

In this paper, we consider that the prey population increases with the Malthusian growth model in the absence of predator, whereas the predator population decreases as the proportion of its population size. We consider the killing rate of prey attacking by predator, and include the interaction rate on the system of ordinary differential equation. Further, we present the model equation with harvesting effort as the proportional to their population size. We analyze the population dynamic by doing the variation of harvesting efforts rate. We also consider the harvesting efforts rate is same for both species. We perform here a careful sensitivity analysis in different harvesting efforts rate of two species. This would substantially help in the planning of conservations of the two species in the certain bounded region.

Formulation of model equation

Let $x(t)$ and $y(t)$ be the number of prey (for example deer) and the number of predator (for example tiger) at time t, respectively. If there is an unlimited food supply for prey species, then in the absence of predator the growth of the prey is represented by the equation (Malthus, 1798):

$$\frac{dx}{dt} = \alpha\, x(t), \quad \alpha > 0, \qquad \ldots(1)$$

Where α is the difference between birth and death rate. If there were no food supply, then the predator population would die out at a rate proportional to its population growth as:

$$\frac{dy}{dt} = -\beta\, y(t), \quad \beta > 0, \qquad \ldots(2)$$

Where β is the die out rate of the predator. If the rate at which predator population encounters the prey population is jointly proportional to the sizes of the two populations, then the system of ordinary differential model equation (Lotka, 1925; Voltera, 1938) as:

$$\frac{dx}{dt} = \alpha\, x(t) - \gamma\, x(t)\, y(t), \quad \alpha > 0, \gamma > 0, \qquad \ldots(3)$$

$$\frac{dy}{dt} = -\beta\, y(t) + \delta\, x(t)\, y(t), \quad \beta > 0, \delta > 0, \qquad \ldots(4)$$

Where γ and δ are the attraction rate of the predator so that the population of prey is decreasing and the population of the predator is increasing. If both species are considered as harvesting, and this is carried out on assuming the demand in the market of both species, then Lotka-Volterra model equation is written in the form:

$$\frac{dx}{dt} = \alpha\, x(t) - \gamma\, x(t)\, y(t) - E\, x, \quad \alpha > 0, \gamma > 0, \qquad \ldots(5)$$

$$\frac{dy}{dt} = -\beta\, y(t) + \delta\, x(t)\, y(t) - E\, y, \quad \beta > 0, \delta > 0, \qquad \ldots(6)$$

where the positive constant E is the harvesting efforts for both species with initial populations are $x(0) = x_0$, and $y(0) = y_0$, and α, β, γ, δ are positive constants. However, the presence of predators also causes the number of deer to decline in proportion to the number of encounters between a tiger and a deer, which is proportional to the product $x(t)\, y(t)$. There are some assumptions that population of predator has only one food source, and it will die in the absence of the specific prey instead of resorting to another food source. The population of tiger reduces in the absence of the population of deer, and there is no other threat to prey population other than one species tiger. The system of model equation (5) and (6) are solved numerically by the method of Runge-Kutta fourth order (Kresyzig, 2009).

The Trajectories

Dividing the equation (6) by (5), we obtain

$$\frac{dy}{dx} = \frac{y(-\beta + \delta x - E)}{x(\alpha - \gamma y - E)}. \qquad \ldots(7)$$

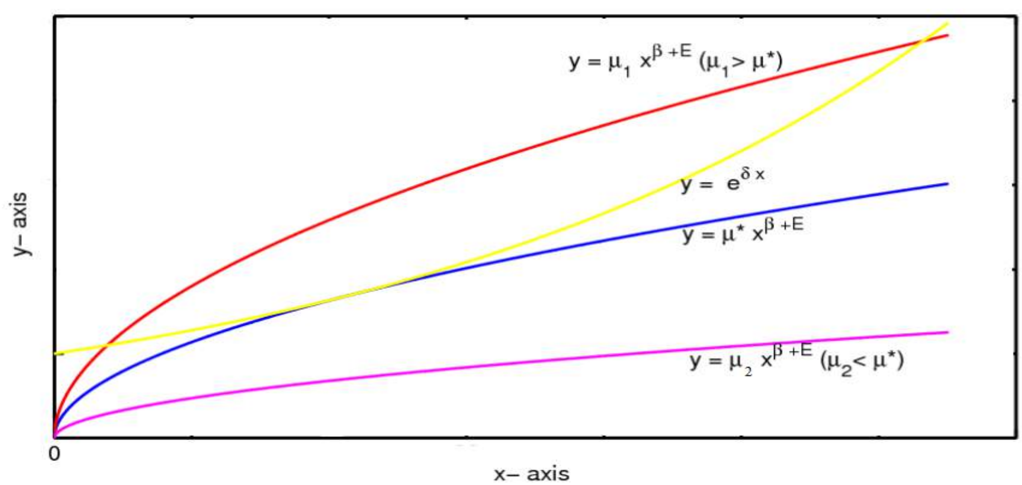

Figure 1.1: Determination of the points of intersection of curves.

Integrating (7), we get

$$(\alpha - E)\log y - \gamma y = \delta x - (\beta + E)\log x + c,$$

where c is constant of integration. Using initial conditions, $x(0) = x_0$, and $y(0) = y_0$, we get

$$(\alpha - E)\log y_0 - \gamma\, y_0 = \delta\, x_0 - (\beta + E)\log x_0 + c, \qquad \ldots(8)$$

$$\therefore\ c = (\alpha - E)\log y_0 - \gamma\, y_0 - \delta\, x_0 + (\beta + E)\log x_0.$$

Substituting the value of c in (8), we get

$$(\alpha - E)\log\left(\frac{y}{y_0}\right) - \gamma\,(y - y_0) = \delta\,(x - x_0) - (\beta + E)\log\left(\frac{x}{x_0}\right),$$

$$\text{or,}\quad \frac{x^{\beta+E}}{e^{\delta x}}\,\frac{e^{\delta x_0}}{x_0^{\beta+E}} = \frac{e^{\gamma y}}{y^{\alpha-E}}\,\frac{y_0^{\alpha-E}}{e^{\gamma y_0}}.$$

Since (x_0, y_0) lies in the first positive quadrant, we get a family curve as:

$$x^{\beta+E}\, e^{-\delta x} = \lambda\, e^{\gamma y}\, y^{-(\alpha-E)}, \quad \text{where } \lambda = x_0^{\beta+E}\, y_0^{\alpha-E}\, e^{-\gamma y_0 - \delta x_0}.$$

These curves are in $x\,y$ - plane which are the trajectories of model equation (5) and (6). The points of intersection of the curves with lines parallel to the coordinate axes, as taking the line parallel to the x- axis is $y = k$, as

$$x^{\beta+E}\; e^{-\delta x} = \lambda\, e^{\gamma k}\, k^{-(\alpha-E)} = \frac{1}{\mu},$$

$$\therefore \;\; F(x) \equiv \mu\, x^{\beta+E} - e^{\delta x} = 0. \qquad ....(9)$$

Differentiating (9) with respect to x, we get

$$F'(x) \equiv \mu\,(\beta+E)\, x^{\beta+E-1} - \delta\, e^{\delta x} = 0. \qquad ....(10)$$

Eliminating x between (9) and (10), we get the critical values of μ and the values of x as

$$\mu = \mu^* = \frac{e^{\beta+E}\,\delta^{\beta+E}}{(\beta+E)^{\beta+E}}, \text{ and } x = x^* = \frac{\beta+E}{\delta}.$$

The roots of (9) are obtained by the abscissa of the points of intersection of the curves,

$$y = e^{\delta x}, \quad \text{and} \quad y = \mu\, x^{\beta+E}. \qquad ...(11)$$

These two curves intersect in two real and equal points if $\mu = \mu^*$, or in two different points if $\mu = \mu_1$ or do not intersect in any points if $\mu = \mu_2$ as shown in Fig. 1.1.

Characteristic Speeds and Critical Values

For the critical points, putting $dx/dt = 0$, $dy/dt = 0$ give

$$x\,(\alpha - \gamma y - E) = 0, \qquad ...(12)$$

$$y\,(-\beta + \delta x - E) = 0. \qquad ...(13)$$

So, the critical values are (0, 0), and $\left(\frac{\beta+E}{\delta}, \frac{\alpha-E}{\gamma}\right)$. In the first case, there are no species no predators and no prey, and this is catastrophic situation. The second is a nontrivial solution in which both populations maintain a steady value, for which the birth rate of the prey is precisely sufficient to continuously feed the predators. The other possible critical values are $\left(\frac{\beta+E}{\delta}, 0\right), \left(0, \frac{\alpha-E}{\gamma}\right)$. These are the critical values from where the population sizes are decreasing or increasing. Both the populations are increasing if first derivatives $dx/dt > 0$, $dy/dt > 0$, and decreasing if $dx/dt < 0$, $dy/dt < 0$. The characteristic speeds are determined from the Eigenvalues of the system. The model equations (5) and (6) can be put in the matrix form as:

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F}(\boldsymbol{X}), \qquad ...(14)$$

where $\boldsymbol{X}$ is the vector of function with and $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{X})$ is the vector of function with:

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{F}(\boldsymbol{X}) = \begin{bmatrix} \alpha x - \gamma x y - E x \\ -\beta y + \delta x y - E y \end{bmatrix}. \qquad ...(15)$$

One of the main tasks here is to obtain the Eigenvalues of the system (14) as characterized by (15). The Eigenvalues are connected with the Jacobian matrix as

$$\boldsymbol{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}} = \begin{bmatrix} \alpha - \gamma y - E & -\gamma x \\ \delta y & -\beta + \delta x - E \end{bmatrix}.$$

Eigenvalues of $\boldsymbol{A}$ are given by the roots of the characteristics polynomial equation $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ in λ at the critical values (0, 0), $\left(\frac{\beta+E}{\delta}, 0\right)\left(0, \frac{\alpha-E}{\gamma}\right)$, and $\left(\frac{\beta+E}{\delta}, \frac{\alpha-E}{\gamma}\right)$. The critical point $\left(\frac{\beta+E}{\delta}, 0\right)$ refers that there are only prey population exist in the absence of predator population, the point $\left(\frac{\beta+E}{\delta}, \frac{\alpha-E}{\gamma}\right)$ means that the populations of two species are coexistence in this equilibrium. In the equilibrium point (0, 0), $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ gives

$$\begin{vmatrix} \alpha - E - \lambda & 0 \\ 0 & -\beta - E - \lambda \end{vmatrix} = 0,$$

$$\therefore \quad \lambda = \alpha - E, \; -(\beta + E).$$

The growth rate of the population of prey is greater than the product of the harvesting rate E with the catchability rate of the prey, and one Eigenvalue is positive and another is negative. Since one Eigenvalue is positive for $\alpha > E$, and another Eigenvalue is negative, then the system is semi-stable. In the equilibrium point $\left(\frac{\beta+E}{\delta},\ 0\right)$, $|\boldsymbol{A}-\lambda \boldsymbol{I}| = 0$ gives

$$\begin{vmatrix} \alpha - E - \lambda & -\dfrac{\gamma\beta+\gamma E}{\delta} \\ 0 & 0-\lambda \end{vmatrix} = 0,$$

$$\therefore \quad \lambda = 0,\ \alpha - E.$$

Since $\alpha > E$, one Eigen value is positive, and another is zero, the system is unstable. In the equilibrium point $\left(\frac{\beta+E}{\delta},\ \frac{\alpha - E}{\gamma}\right)$, $|\boldsymbol{A}-\lambda \boldsymbol{I}| = 0$ gives

$$\begin{vmatrix} 0-\lambda & -\dfrac{\gamma\beta+\gamma E}{\delta} \\ \dfrac{\delta\alpha-\delta E}{\gamma} & 0-\lambda \end{vmatrix} = 0,$$

$$\therefore \quad \lambda = \pm i\sqrt{(\beta+E)(\alpha-E)}.$$

Since $\alpha > E$, both Eigenvalues are imaginary, so the system is asymptotically stable, and its solution will be periodic.

Numerical Results and Discussion

Figure 1.2 reveals that the initially twenty tigers and forty deer are existing in the region; the growth rate of the prey population, $\alpha = 0.56$ per capita, the rate of decreasing of predator population $\beta = 0.64$ per capita, and the interaction rate $\gamma = \delta = 0.012$. Although the harvesting effort for both species are taken $E = 0.0225$, both species are survived periodically a long period of time. One hundred four is the maximum population sizes of prey at $t = 4$ yrs, 15 yrs, 26 yrs, 37 yrs, 40 yrs, and so on. Every eleven years, the populations of deer are periodically maximum as in Fig. 1.2. In the case of predator, ninety is the maximum population sizes of predator at $t = 6$ yrs, 7 yrs, 28 yrs, 39 yrs, 50 yrs and so on. Every eleven years, the populations of tiger are periodically maximum as in Fig. 1.2.

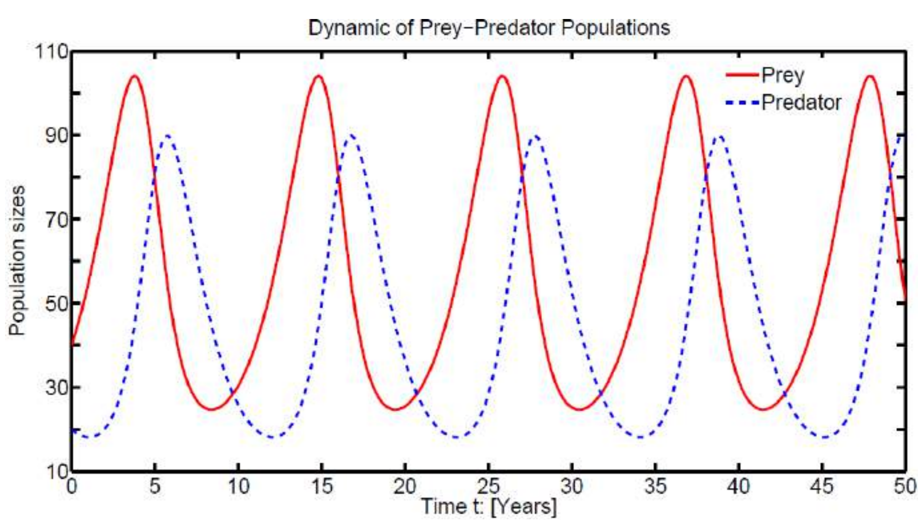

Figure 1.2: The prey-predator populations with harvesting effort $E = 0.0225$.

Figure 1.3 shows that if the harvesting efforts rate is increased by 100% for both species in the model equation, then the maximum number of prey is increased by 0.961% and reached up to 105 prey population sizes whereas the maximum number of predator is decreased by 5.5% and reached up to 85 predator population. Both the populations are existing periodically a long period of time.

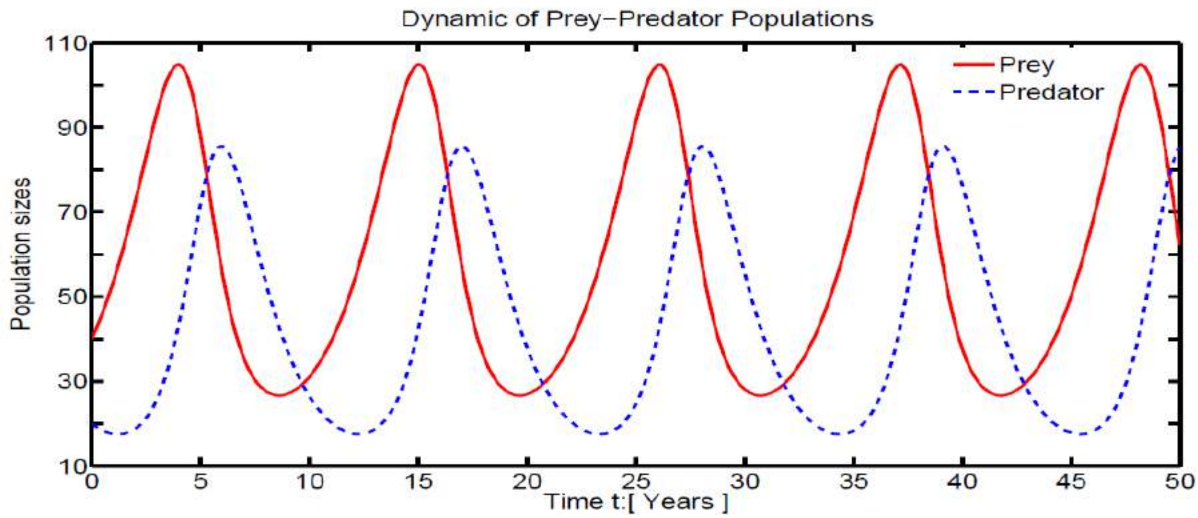

Figure 1.3: The prey-predator populations with harvesting effort $E = 0.0450$.

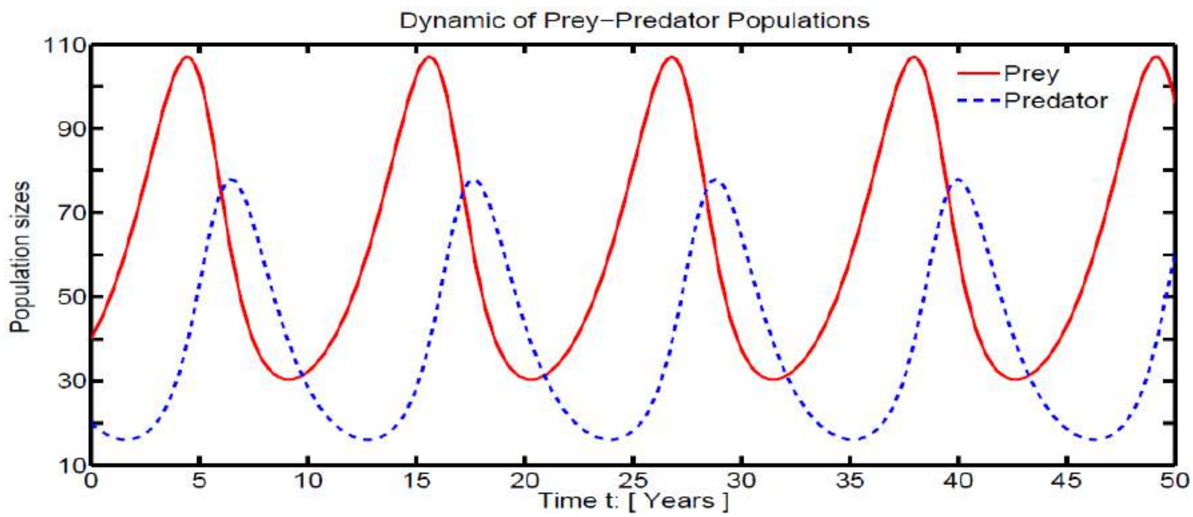

Figure 1.4: The prey-predator populations with harvesting effort $E = 0.0900$.

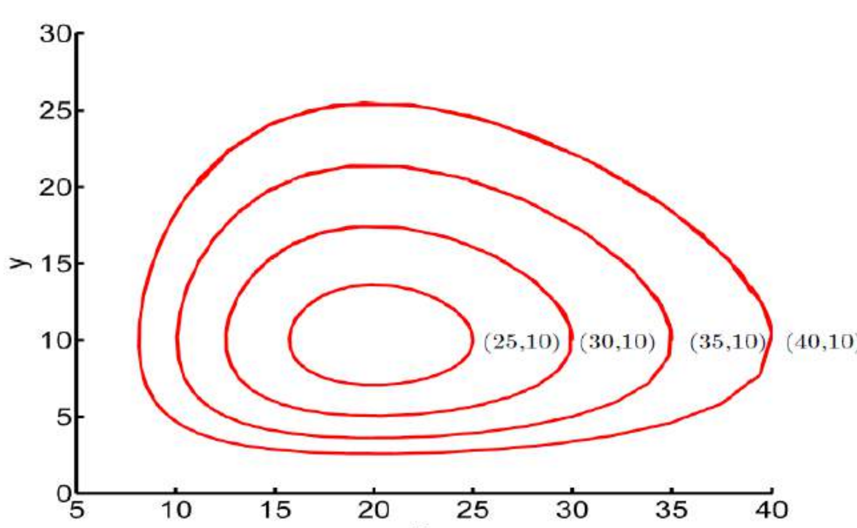

Figure 1.5: The trajectories of the system of prey-predator model with harvesting.

Figure 1.4 reveals that if the harvesting efforts rate is increased by 100% for both species in the model equation, then the maximum number of prey is increased by 1.9048% and reached up to 107 prey population sizes whereas the maximum number of predator is decreased by 8.2353 % and reached up to 78 predator population. Both the populations also exist periodically a long period of time. The trajectories of the population of both species are shown in Fig. 1.5.

Conclusion

The numerical solution of the prey predator model with harvesting was obtained by using Runge-Kutta fourth order method. In the system, the harvesting effort rates are the same for both species. We observed that the populations of both species are survived periodically for a long period of time if the harvesting efforts rates are the same for both species. In this case, we found that at the certain period of time, both prey and predator population increases. The prey population increases by less percentage whereas the predator population increases significantly by high percentage. The qualitative dynamics of population depended on the harvesting efforts rate in which the populations of prey species are slowly increasing, and the population of predator is decreasing but they are coexisting for a long period of time; the results showed that the population dynamics depend on the initial population, growth rate, interaction rate and also harvesting efforts. The dynamic of the system are also observed by taking different harvesting efforts rate for both species in the certain bounded region. This would be useful for analyzing the population dynamic of various types of species within the same compartment.

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