Generalization of Lagrange's Mean Value Theorem

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Abstract : This paper deals with the mean value theorem of differential calculus due to the Lagrange's mean value theorem which has appeared in Lagrange's book Theories functions analytiques in 1797 as an extension of Rolle's result from 1619. We state and prove a theorem which is similar to the generalized mean value theorem for the single variable function. Firstly, we review the mean value theorems. Moreover, the theorem is illustrated with some examples. The main propose of this paper is to establish the different forms of Lagrange's Mean Value Theorem, as an extended forms.

Key Words: Rolle's mean value theorem, Lagrange's mean value theorem, Cauchy mean value theorem, Extended Lagrange's mean value theorem.

1. Introduction

Mean value theorem plays a vital role in mathematics as well as in engineering. It calculates the rate of change of continuous functions. It helps to develop the theoretical mathematics as well as applied mathematics. Mean value theorem of differential and integral calculus provides a relatively simple , but very powerful tools of mathematical analysis suitable for solving many diverse problems. In 1999, P. K. Sahoo and et al. [6] introduced the first mean value theorem for homorphic function and then the area of calculus is more expanded.

Mean value theorem deals the function and its slope of tangent at a point in the given interval. In other word the theorem studies about the rate of change of continuous function under the fixed interval. The extended form of Lagrange's mean value theorem i.e. the modern form of mean value theorem was presented by a French mathematician **Augustin Louis Cauchy** (1798-1857) in 1823 [2]. The extended version deals the mean of double functions. This helps to turn the calculus. An analogous of the theorem due to a German mathematician **Karl Theodor Wilhelem Weierstrass** (1815-1897) which now known as second mean value theorem for integration. After then mean value theorem leads the calculus [5]. Later, many mathematicians gave different analogous results on different fields as differential calculus, integral calculus , vector calculus and other many branches of mathematics [3].

2. Some CommonTheorems on Mean Value Theorem

Without giving the detail proof, here we state some common mean value theorems, which are frequently used in different forms of calculus and their generalized forms. We will use these theorems to prove the result in this paper.

2.1 Rolle's Theorem *: If a real valued function f(x) continuous on closed interval [a, b] and differentiable on open interval (a ,b) also f(a) =f(b) then there exists a point c in (a, b) such that f'(c) =0*

Some times in some functions may give the value of c in (a, b) but do not satisfy all the conditions of Rolle's theorem. Such as if $f(x)= x^2-x$ for x in (0,2) then clearly it satisfies first two conditions but $f(0)\neq f(2)$, although c=0.5 in(0,2).

If the third condition in the above theorem, is ignored then f'(x) is equal to the slope of the curve f(x) joining the points (a, f(a)) and (b, f(b)). This is known as the Lagrange's mean value theorem and due to **Joseph Louis Langrage.**

Proof:

Let c=f(a)=f(b). We consider three cases:

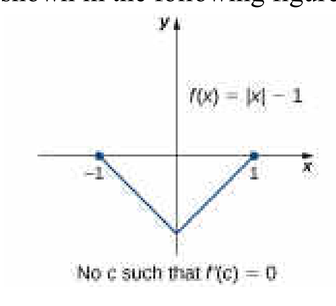
1. f(x)=c for all x∈(a, b).
2. There exists x∈(a ,b) such that f(x) > c.
3. There exists x∈(a, b) such that f(x) < c.

Case 1: If f(x)=0, for all x∈(a, b) then f′(x)=0.

Case 2: Since f is a continuous function over the closed, bounded interval [a, b] by the extreme value theorem it has an absolute maximum.

Also, since there is a point x∈(a, b) such that f(x)> c , the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point c∈(a, b). Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, f′(c)=0.

If f is not differentiable, even at a single point, the result may not hold. For example [1] the function f(x)=|x|−1 is continuous over [−1,1] and f(−1)=0=f(1) but f′(c)≠0 for any c∈(−1,1)c∈(−1,1) as shown in the following figure.



Since f(x)=|x|−1 is not differentiable

at $x=0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c\in(-1,1)$ such that $f'(c)=0$..

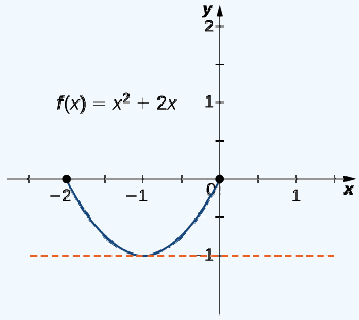
Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c)=0.f'(c)=0$.

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c)=0$.

a. $f(x)=x^2+2x$ over $[-2,0]$
b. $f(x)=x^3-4x$ over $[-2,2]$

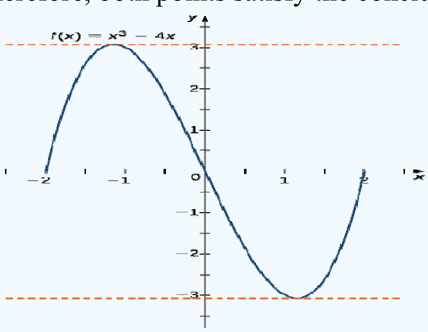
Solution

Since f is a polynomial, it is continuous and differentiable every where. In addition, $f(-2)=0=f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c\in(-2,0)$ such that $f'(c)=0$. Since $f'(x)=2x+2=2(x+1)$, we see that $f'(c)=2(c+1)=0$ implies $c=-1$ which is shown in the following graph.



This function is continuous and differentiable over $[-2,0]$, $f'(c)=0$, when $c=-1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2)=0$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x)=3x^2-4$. Therefore, $f'(c)=0$. Both points are in the interval $[-2,2]$ and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.



For this polynomial over $[-2, 2]$, $f'(c)=0$, at $x=\pm2/3$

2.2 Lagrange's theorem : *If a function f(x) is continuous on closed interval [a ,b], differential on open interval (a ,b) then there exists at least one point c in (a ,b) such that f'(c) =*$\frac{f(b)-f(a)}{b-a}$

If the above theorem consists of another function g(x) under the same conditions as f(x) then it gives the another result which is due to as **Augustin Louis Cauchy [4].**

2.3 Cauchy's Theorem: *If f(x) and g(x) are any two functions which are continuous on closed interval [a, b], differential on open interval (a ,b) then there exists at least one point c in (a, b) such that* $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ *provided that g'(c)≠0*

Which is the generalization form of Lagrange's theorem .

By inserting (b-a) on the RHS of Cauchy Mean Value Theorem as the quotient, one for f and another for g. Then we can see that The Lagrange's MVT gives a trivial proof. But there may be an error. The fixed point c for f(x) by Lagrange's MVT may not be same for g(x) by Cauchy MVT theorem. So that the fixed value c for Lagrange MVT may quite different for Cauchy MVT. So we can't apply the Lagrange MVT in the proof of Cauchy MVT.

If we consider the following example:

Example 2.4: Let f(x)=x^2 - 2x then Lagrange's mean value theorem gives c=1.16 ∈(0,2) and if we define another function on the same interval as g(x) = x^2 then by Cauchy mean value theorem gives c=1.72 ∈ (0,2), which are not same.

3. Main Result

The main result is that the Common Mean Value Theorem is the extended form of Lagrange's mean value theorem in which the resultant value is in quotient form. Here the resultant values are arranged in the linear form with unit constant value. The following theorem is also the extended form of Lagrange's mean value theorem.

3.1 Theorem (Extended Lagrange's Mean Value Theorem): *If f(x) and g(x) are any two functions which are continuous on closed interval [a , b] and differential on open interval (a, b) then there exists at least one point c∈(a,b) such that*

$$f'(c) + g'(c) = \frac{f(b)-f(a)}{b-a} + \frac{g(b)-g(a)}{b-a}$$

proof: let the functions are continuous on closed interval [a , b] and differentiable on open interval (a , b).

Let us define a function as ø(x)=[f(x)+g(x)] (b-a)-[f(b)-f(a)]x-[g(b)-g(a)]x, which is continuous on [a, b] and differential on (a ,b)

Now,

Ø(a)= [f(a) +g(a)](b –a) –[f(b) –f(a)]a –[g(b)- g(a)]a

=f(a)b- f(a)a +g(a)b –g(a)a –f(b)a +f(a)a –g(b)a +g(a)a

=f(a)b +g(a)b –f(b)a –g(b)a

And
Ø(b) =[f(b) +g(b)](b –a) –[f(b)-f(a)]b –[g(b) –g(a)]b
=f(b)b –f(b)a +g(b)b –g(b)a –f(b)b +f(a)b –g(b)b +g(a)a
=f(a)b +g(a)b –f(b)a –g(b)a
=ø(a)
Hence ø(x) satisfies all the conditions of Rolle's theorem. So by Rolle's theorem there exists a point c in (a ,b) such that ø'(c) =0
Since,
Ø(x) =[f(x) + g(x)](b-a) –[f(b)-f(a)]x –[g(b)- g(a)]x
Therefore,
0 = ø'(c) = [f'(c) +g'(c)](b- a) –[f(b)-f(a)] –[g(b) –g(a)]
So, f'(c) +g'(c) $=\frac{f(b)-f(a)}{b-a}+\frac{g(b)-g(a)}{b-a}$
Which gives the complete proof.
Note: If we choose g(x)= x or g(x)= c in the theorem then it reduces to the Lagrange's mean value theorem.
Remark: If f'(x) =0 and g'(x) =0 for all x in (a, b) then f(x) and g(x) are constant and the theorem gives f(b) –f(a) = g(b) –g(a) that is the chords are parallel.

Example 3.2: If f(x) $=x^2$ and g(x)= x^2-2x for x in [0,2] then from this example extended Lagrange's mean value theorem can be verified but this does not verify Cauchy mean value theorem.

Example 3.3: Let f(x) = 3x-2 for x in [1, 2] then by Lagrange's mean value theorem, we observe that c$=\frac{3}{2}\in(1,2)$. We extend the above theorem to verify the Cauchy mean value theorem.

Example 3.4: Let f(x) $=x^2$ and g(x)= 3x -2 for x in [1,2] then by Cauchy mean value theorem we see that c$=\frac{3}{2}\in(1,2)$.

Example 3.5: Let f(x) = x and g(x) $=x^2$-2 for x ∈ [0 , 2] . We get g'(1) =0 , at x =1 .
Which shows that this example does not satisfies the Cauchy mean value theorem.
However, this example is useful to verify the extended Lagrange's mean value theorem.
Also we can establish another theorem from the above theorem.

Theorem 3.6: *If f(x) and g(x) are any two functions , which are continuous on closed interval [a ,b] and differentiable on open interval (a ,b) then there exists at least one point can (a ,b) such that f'(c) –g'(c)* $=\frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}}-\frac{\boldsymbol{g(b)-g(a)}}{\boldsymbol{b-a}}$

This theorem can also be verified by the above example 2

Theorem 3.7: *If* $n\in\mathbb{N}$ *and assume that f(x),* $g_1(x)$, $g_2(x)$, $g_3(x),\ldots\ldots g_n(x)$ *be the n+1 continuous functions defined on the closed and bounded interval [a, b] and are differentiable on* *open interval (a, b) and* $g_i'(x) \neq 0\ \forall\ x \in (a, b)$ *where i=1, 2, 3,……n. Then there exists* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{n} [\sum_{i=1}^{n} \frac{g'_{i(c)}}{g_i(b)-g_i(a)}]$.

Proof: Let us define a function $G(x) = n(f(x)-f(a)) - \sum_{i=1}^{n} \frac{f(b)-f(a)}{g_i(b)-g_i(a)} (g_i(x)- g_i(a))$.

As $g'(x) \neq 0$, which gives $g_i(b) - g_i(a) \neq 0$, for i=1, 2, 3,……n. Since n is finite, so G(x) is well defined on the closed interval [a, b]. Moreover f(a) = f(b) = 0 and G(x) is continuous on closed and bounded interval [a ,b]. Then by algebra of continuous as well as differentiable function [2], it follows that G(x) is differentiable on (a ,b). So the conditions of Rolle's theorem are satisfied, so there exists a point $c \in (a,b)$ such that F'(c) = 0. Which gives

$$Nf'(c) - \sum_{i=1}^{n} \frac{f(b)- f(a)}{g_i(b)- g_i(a)} g'(c)= 0.$$

Thus there exists

$c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{n} [\sum_{i=1}^{n} \frac{g'_{i(c)}}{g_i(b)-g_i(a)}]$.

Which gives the complete proof.

This theorem can be verified by using the following example

Let $f(x) = x + 1$, $g_1(x) = x^2 +4x -4$ and $g_2(x) = x^2 +3x$ defined on the interval [0, 3].

Then these functions are continuous on [0 ,3] and differentiable on [0, 3] with $g'_i(x) \neq 0\ \forall\ x \in (0, 3)$ for i= 1,2. So the conditions are satisfied, hence $\exists\ c \in (0, 3)$:

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Which gives 78c=117 and so $c=1.5 \in (0, 3)$ as required.

uitable for solving

Similarly the extended Lagrange's mean value theorem is also true for the product of any two functions under the same conditions of Lagrange's mean value theorem.

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