Journal of Bhuwanishankar (JoBS)
A Peer Reviewed Research Journal
ISSN: 2961-1938 (Print), 2961-1946 (Online)

q-Analogue of Hermite-Hadamard Type Inequalities for s-Convex Functions in the Breckner Sense

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Received: March 20, 2022 Accepted: July 25, 2022
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Abstract:
Hermite and Hadamard independently introduced the Herimite-Hadamard inequality for convex functions for the first time. In recent years, a variety of extensions have been made with the use of convex functions by several researchers. In this paper, we have given a variant of the Hermite-Hadamard integral inequality for the s-convex function in the Breckner sense.

Key Words: Hermite-Hadamard Inequality, s-convex functions, q-derivative, Jackson q- in- tegration.

1. Introduction and preliminaries

In mathematics, quantum calculus is the study of classical calculus without the notation of limit, and it is also known as q-calculus, where q is a parameter $0 < q < 1$. In q - calculus, we obtain mathematical expression in terms of q and whenever $q \to 0$, it again reduces to the original form. The history of the q-calculus can be traced back to Euler (1707- 1983), who first introduced theq-calculus to deal with Newton's work of infinite series. In the twentieth century, Jackson [2] was the first mathematician, who started the systematic study of q- calculus and introduced the q-definite integral [8]. In 1893, Hermite-Hadamard investigated one of the fundamental inequalities in analysis, that is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1)$$

which is known as the Hermite-Hadamard inequality. For the first time, in 2014, Tariboon and Ntouyas [4] investigated the q-analogue of several of classical integral inequalities, from which they obtained the q-analogue of the Hermite - Hadamard inequality. But their finding was not compatible for $q \in (0, 1)$ for the left-hand side, which was proved in 2016 Alp. et al.[5] by giving a counter example and proving the correct q- Hermite Hadamard inequality. Recently, many extensions have been given with the use of convex functions by several researchers. The investigation into the q-Hermite -Hadamard inequality for general convex functions can be found in 2020 [6].

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The variant of the Hermite- Hadamard result for s-convex functions in the second sense or Breckner sense is

$$2^{s-1} f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{2}$$

As $s = 1$ it reduces to (1). The purpose of this paper is to present the q- calculus of Hermite-Hadamard inequalities for s-convex function in the Breckner's sense

We now present some notations and definitions from the q-calculus, which are necessary for understanding this paper. Let $J := [a, b] \subset \mathbb{R}$ be an interval and q be a constant with $0 < q < 1$.

Definition 1. *[3] The q-derivative of a continuous function $f : J \to \mathbb{R}$ at x is defined as:*

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)} \; for \; x \neq a \tag{3}$$

For $x = a$ it is defined as

$${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x)$$

If ${}_aD_q f(x)$ exists for all $x \in J$, then f is q- differentiable on J. Moreover, if $a = 0$, then (5) reduces to

$${}_0D_q f(x) = D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}; \; x \neq 0$$

For more details, see [8]

The higher -order q-derivatives of functions on J are also defined.

Definition 2. *[3] For a continuous function $f : J \to \mathbb{R}$, the second - order derivative of f on J, if ${}_aD_q f$ is q- differentiable on J , denoted by ${}_aD_q^2 f$ and defined by*

$${}_aD_q^2 f = {}_aD_q({}_aD_q) f$$

Similarly, n^{th} order q- derivative ${}_aD_q^n f$ can be defined on J , provided that ${}_aD_q^{n-1} f$ is defined on J.

Definition 3. *[3] Let $f : J \to \mathbb{R}$ be a continuous function. Then the q-definite integral on J is represented as*

$$\int_a^x f(t) \; {}_ad_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q)a) \; ; for \; x \in J. \tag{4}$$

If a=0 in (6) , it reduces to the classical q-integral called Jackson's q-integral on [0, x] delineated [8] as

$$\int_0^x f(t) \; {}_0d_q t = \int_0^x f(t) \; d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n) \; ; for \; x \in [0, \infty) \tag{5}$$

Theorem 1. *Assume that function $f : j \to \mathbb{R}$ is continuous. Then, we have the following*

(i) ${}_aD_q \int_a^x f(t)\ {}_ad_qt = f(x) - f(a)$;

(ii) $\int_c^x\ {}_aD_qf(t)\ {}_ad_qt = f(x) - f(c)$ *for* $c \in (a, x)$

Theorem 2. *Assume that function $f, g : j \to \mathbb{R}$ is continuous be a continuous function and $k \in \mathbb{R}$. Then we have the following*

(i) $\int_a^x [f(t) + g(t)]\ {}_ad_qt = \int_a^x f(t)\ {}_ad_qt + \int_a^x g(t)\ {}_ad_qt$;

(ii) $\int_a^x (kf)(t)\ {}_ad_qt = k \int_a^x f(t)\ {}_ad_qt$;

(iii) $\int_a^x f(t)\ {}_aD_qg(t)\ {}_ad_qt = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\ {}_aD_qf(t)\ {}_ad_qt$ *for* $c \in (a, x)$

The proof of fundamental theorem on integral calculus , linear property and integration parts in Theorems (1) and (2), see [3]

Using lemma 2

Definition 4. *[4] For $\alpha \in \mathbb{R} - \{-1\}$, the definite q- integral is given by*

$$\int_a^x (t-a)^\alpha\ {}_ad_qt = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1} \tag{6}$$

From this one can write

$$\int_0^x t^\alpha\ {}_0d_qt = \left(\frac{1-q}{1-q^{\alpha+1}}\right)x^{\alpha+1} \tag{7}$$

Definition 5. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be s- convex function in the second sense or convex in the Breckner sense if*

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v) \tag{8}$$

for all $u, v \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and s fixed in (0, 1]

The set of all s-convex functions in the second sense is denoted by K_s^2.

2. q-analogue of Hermite-Hadamard inequality for s-convex functions in Breckner sense

Theorem 3. *Let $0 \leq a < b < \infty$ and $J := [a, b]$ and $0 < q < 1$ be a constant. Let f be s-convex function in the second sense or s -convex in the Breckner sense, then q- Hermite -Hadamard inequality variant is given by*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\ {}_ad_qx \leq \Big(f(a) + f(b)\Big)\frac{1-q}{1-q^{s+1}} \tag{9}$$

Proof. As f is s-convex function in the second sense, for all $f \in [0,1]$ we have

$$f(ta+(1-t)b) \leq t^s f(a) + (1-t)^s f(b) \tag{10}$$

q-Integrating over [0, 1], we get

$$\int_0^1 f(ta+(1-t)b)\ {}_0d_qt \leq f(a)\int_0^1 t^s\ {}_0d_qt + f(b)\int_0^1 (1-t)^s\ {}_0d_qt \tag{11}$$

Now,

$$\begin{aligned}
\int_0^1 f(ta+(1-t)b)\ {}_0d_qt &= (1-q)(1-0)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b + (1-q^n)\cdot 0) \\
&= (1-q)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) \\
&= \frac{(1-q)(b-a)}{(b-a)}\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) \\
&= \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx
\end{aligned}$$

$$\therefore \int_0^1 f(ta+(1-t)b)\ {}_0d_qt = \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \tag{12}$$

We have

$$\int_0^1 f(x)\ {}_0d_qx = (1-q)\sum_{n=0}^{\infty} q^n f(q^n)$$

So,

$$\int_0^1 t^s\ {}_0d_qx = (1-q)\sum_{n=0}^{\infty} q^n (q^n)^s \tag{13}$$

$$= (1-q)\sum_{n=0}^{\infty} q^{n(s+1)} \tag{14}$$

$$= \frac{1-q}{1-q^{(s+1)}} \tag{15}$$

Again,

$$\int_0^1 (1-t)^s\ {}_0d_qt$$

Let, 1- t = y

$$D_q(1-t) = D_q y$$
$$d_q t = -d_q y$$

As , t= 0 , then y=1 and as t= 1 , then y = 0.
So,

$$\begin{aligned}\int_0^1 (1-t)^s \, {}_0d_q t &= -\int_1^0 y^s \, {}_0d_q y \\ &= \int_0^1 y^s \, {}_0d_q y \\ &= \frac{1-q}{1-q^{s+1}} \end{aligned} \tag{16}$$

From (11),(12), (15) and (16) we get,

$$\begin{aligned}\int_0^1 f(ta+(1-t)b) \, {}_0d_q t &\le f(a)\int_0^1 t^s \, {}_0d_q t + f(b)\int_0^1 (1-t)^s \, {}_0d_q t \\ \frac{1}{b-a}\int_a^b f(x) \, {}_ad_q x &\le f(a)\frac{1-q}{1-q^{s+1}} + f(b)\frac{1-q}{1-q^{s+1}} \\ \therefore \frac{1}{b-a}\int_a^b f(x) \, {}_ad_q x &\le \Big(f(a)+f(b)\Big)\frac{1-q}{1-q^{s+1}} \end{aligned} \tag{17}$$

Again for the second part. Let $x, y \in I$. Since $f \in K_s^2$, for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ we have

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2^s} \tag{18}$$

Without loss of generality we may assume that

$$x = (1-t)a + tb$$
$$y = ta + (1-t)b$$

From (18)

$$\begin{aligned} f\Big(\frac{(1-t)a+tb+ta+(1-t)b}{2}\Big) &\le \frac{1}{2^s}\Big[f((1-t)a+tb) + f(ta+(1-t)b)\Big] \\ f\Big(\frac{a+b}{2}\Big) &\le \frac{1}{2^s}\Big[f((1-t)a+tb) + f(ta+(1-t)b)\Big] \end{aligned}$$

q-integrating over [0, 1]

$$\int_0^1 f\Big(\frac{a+b}{2}\Big)\ {}_od_qt \leq \frac{1}{2^s}\Big[\int_0^1 f((1-t)a+tb)\ {}_od_qt + \int_0^1 f(ta+(1-t)b)\ {}_od_qt\Big]$$

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{2^s}\Big[\frac{1}{(b-a)}\int_a^b f(x)\ {}_ad_qx + \frac{1}{(b-a)}\int_a^b f(x)\ {}_ad_qt\Big]$$

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{2^s}\frac{2}{(b-a)}\int_a^b f(x)\ {}_ad_qx$$

$$2^{s-1}f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)}\int_a^b f(x)\ {}_ad_qx \tag{19}$$

Using (17) and (19)

$$2^{s-1}f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \leq \Big(f(a)+f(b)\Big)\frac{1-q}{1-q^{s+1}} \tag{20}$$

Remarks: As $q\to1$ and $s\to 1$, then (20) reduces to (1).

3. Conclusion

In this paper, we have given the quantum calculus version of the S. Bermudo et. al. (2020), our inequality (20) reduces to the inequality (2) as $q \to 1$ and further if $s \to 1$ it reduces to the classical Hermite-Hadamard type inequality.

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