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Delta Power Transformation: A New Family of Probability Distributions

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Abstract: *This research article presents a novel approach called the delta-power transformation, which introduces a new class of lifetime distributions. The article discusses the key features of one member from this family, which exhibits a hazard function with a distinctive shape resembling a J, reverse-J, constant, or monotonically increasing. The researchers delve into the statistical characteristics of this distribution and utilize the maximum likelihood estimation (MLE) approach to gauge its parameters. They conduct a simulation experiment to evaluate the precision of the estimation process, finding that biases and mean square errors diminish with larger sample sizes, even in cases involving small samples. Moreover, the study showcases the practical utility of the suggested distribution through an examination of two real-world datasets. Evaluation criteria for model selection and goodness-of-fit test statistics indicate that the proposed model surpasses certain existing models in performance.*

Keywords: Power transformation, Weibull distribution, Moment, Cramer-Rao inequality

1 Introduction

In practical applications, statistical models are widely employed to represent and analyze datasets. Traditional distributions such as Weibull, gamma, Lomax, beta, log-normal, and exponential are commonly used for this purpose. However, these distributions may not always provide a satisfactory fit for complex datasets. Consequently, researchers are continuously working on developing new models that are more generalized versions of the existing ones. These advancements often involve techniques like exponentiation and the T-X approach, which aim to generate more adaptable distributions. Recently, Mahdavi and Kundu [13] introduced an alternative approach called the alpha power transformation (APT) family, which incorporates a high degree of skewness and flexibility to the base distribution. They specifically studied the alpha power exponential as a member of this family. Since then, this approach has gained popularity among researchers in the fields of probability theory and survival analysis. Using the APT technique, several authors have proposed new generalized models and families of distributions. For instance, Nassar et al. [20] employed the APT technique to define the new family of distributions using log transformation. Mead et al. [18] have further studied the APT family by providing some mathematical properties that were not provided in [13]. Further, Lomax distribution was transformed using APT by Maruthan and Venkatachalam [16]. Ihtisham et al. [7] and Ihtisham et al. [8] studied the Pareto and inverse Pareto distributions using the APT approach, with the inverse Pareto distribution being applied to model real data related to extreme values. Similarly, Hozaien et al. [6] and Klakattawi and Aljuhani [10] introduced new models using the APT family of distributions. Alotaibi et al. [2] have introduced a new distribution as a weighted form of the APT method while Gomma et al. [4] introduced the alpha power of the power Ailamujia distribution, which offers a flexible hazard function. They utilized this distribution to model COVID-19 datasets from Italy and the UK. Furthermore, Nassar et al. [19] introduced a new family utilizing the quantile function of the APT family whose cumulative distribution function (CDF) is

$$F(t) = \frac{\log\left[1+(\alpha-1)\,G\left(t;\phi\right)\right]}{\log\left(\alpha\right)}; \quad t>0, \alpha>0, \alpha\neq 1,$$

where $G\left(t;\phi\right)$ is the CDF and ϕ is the parameter space of base distribution. Similarly, Elbatal et al. [3] introduced another new APT family whose CDF is

$$F\left(x\right) = \frac{G\left(x\right)\alpha^{G(x)}}{\alpha}; \alpha>0, x\in\Re.$$

As a particular member, new APT-Weibull distribution has been studied. Also using the APT method, Mandouch et al. [14] have reported a new two-parameter family of distributions whose CDF is

$$F(x) = \frac{\alpha^{kW\{G(x)\}} - 1}{\alpha - 1}; \alpha > 0, \alpha \neq 1, x \in \Re.$$

Lone and Jan [12] have introduced another new family using the concept of the APT family and named it the Pi-Exponentiated transformed (PET) family whose CDF is

$$F(x) = \frac{\pi^{\{G(x)\}^{\alpha}} - 1}{\pi - 1}; \alpha > 0, x \in \Re.$$

Hence, researchers are continuously developing and exploring new models and families of distributions to capture the characteristics of complex datasets better. The alpha power transformation (APT) family has emerged as a popular approach, offering increased skewness and flexibility to the base distribution. These advancements have led to the proposal of various generalized models and distributions, which have been successfully applied to a range of datasets, including those related to COVID-19 and extreme values. Building upon the concept of the alpha power transformation (APT), we have introduced a novel method to enhance existing distributions by incorporating an additional parameter, which we refer to as the delta power transformation (DPT) family of distributions. This new family offers increased robustness compared to other compound probability distributions and demonstrates great potential for modeling real-life datasets. The DPT family possesses an extra parameter that enables it to capture a broader range of characteristics exhibited by a dataset, including skewness, kurtosis, and failure rate. This enhanced flexibility allows for a more accurate representation of complex data patterns and distributional properties. By considering the DPT family, researchers and practitioners can better account for the intricate nature of real-world datasets, leading to improved modeling outcomes. Among the members of the DPT family, one distribution stands out as particularly noteworthy - the Weibull distribution. The Weibull distribution has long been employed in reliability theory and life testing due to its ability to capture failure rates and survival probabilities effectively, for more detail see [15]. With the integration of the DPT framework, the Weibull distribution can be further adapted and refined to better align with the unique characteristics observed in various applications.

2 DPT Family and Some Important Functions

Let $X \sim DPT - G$ family, then the CDF and PDF of DPT family $M(y; \delta, \psi)$ and $m(y; \delta, \psi)$ for $y \in \Re$, are defined as

$$M(y;\delta,\psi) = \begin{cases} \frac{\delta}{\delta-1}\left(1 - \delta^{-K(y;\psi)}\right) & \text{for} \quad \psi > 0, \delta > 1, y \in \Re \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$m(y;\delta,\psi) = \begin{cases} \frac{\delta \log \delta}{\delta-1}\delta^{-K(y;\psi)}k(y,\psi) & \text{for} \quad \psi > 0, \delta > 1, y \in \Re \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where $K(y;\psi)$ and $k(y;\psi)$ are the CDF and PDF of a baseline distribution with parameter space ψ. Further reliability, hazard, and quantile functions of the DPT family can be expressed as

$$R(y;\delta,\psi) = \begin{cases} 1 - \frac{\delta}{\delta-1}\left(1 - \delta^{-K(y;\psi)}\right) & \text{for} \quad \delta > 1, y \in \Re \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$h(y;\delta,\psi) = \begin{cases} \frac{(\log\delta)\delta^{1-K(y;\psi)}k(y;\psi)}{\delta^{1-K(y;\psi)}-1} & \text{for} \quad \delta > 1, y \in \Re \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$Q(p) = K^{-1}\left[-\frac{1}{\log\alpha}\log\left(1 - \frac{(\alpha-1)p}{\alpha}\right)\right]. \tag{5}$$

2.1 Some special cases of DPT family

1. When $\delta = e$, in Equation (1) the CDF of DPT tends to KM-Transformation, for more detail see Kavya and Manoharan [9].
2. When $\delta = \alpha^{-1}$ in Equation (1) the CDF of DPT tends to APT, for more detail see Mahdavi and Kundu [13].
3. When $\delta = \pi^{-1}$ in Equation (1) the CDF of DPT tends to PET, for more detail see Lone and Jan [12].

2.2 A linear form of the DPT family

Through some mathematical calculations, the PDF (2) of the DPT family can be expressed in linear form as

$$m(y;\delta,\psi) = \frac{\delta}{\delta-1}\sum_{i=1}^{\infty}\Delta_i K^i(y;\psi)k(y;\psi), \tag{6}$$

where $\Delta_i = (-1)^i\,\frac{(\log\delta)^{1+i}}{i!}$.

3 DPT Weibull (DPTW) Distribution

Let Y be a continuous random variable following the Weibull distribution, then the CDF and PDF are

$$K(y;\psi) = 1 - e^{-\lambda y^{\beta}}; \quad (\lambda,\beta) > 0, y > 0. \tag{7}$$

$$k(y;\psi) = \lambda\beta y^{\beta-1}e^{-\lambda y^{\beta}}; \quad (\lambda,\beta) > 0, y > 0. \tag{8}$$

Now using Equation (7) as a base distribution, we have introduced the DPTW distribution having CDF

$$M(y;\delta,\beta,\lambda) = \frac{\delta}{\delta-1}\left(1-\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}\right) \quad \text{for} \quad \delta > 1, (\beta,\lambda) > 0, y > 0. \tag{9}$$

The PDF of the DPTW distribution can be expressed as

$$m(y;\delta,\beta,\lambda) = \frac{\delta\beta\lambda(\log\delta)}{\delta-1}\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}y^{\beta-1}e^{-\lambda y^{\beta}} \quad ; \quad \delta > 1, y > 0. \tag{10}$$

Further, some key functions such as reliability, hazard, and quantile of DPTW distribution can be presented as

$$R(y;\delta,\beta,\lambda) = 1 - \frac{\delta}{\delta-1}\left(1-\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}\right); \quad y > 0. \tag{11}$$

$$h(y;\delta,\beta,\lambda) = \frac{\beta\lambda(\log\delta)\delta^{e^{-\lambda y^{\beta}}}y^{\beta-1}e^{-\lambda y^{\beta}}}{\delta^{e^{-\lambda y^{\beta}}}-1}; \quad y > 0. \tag{12}$$

$$Q_Y(p) = \left[-\frac{1}{\lambda}\log\left\{1+\left\{\frac{1}{\log\delta}\log\left(1-\frac{(\delta-1)\,p}{\delta}\right)\right\}\right\}\right]^{1/\beta}. \tag{13}$$

and random numbers deviate

$$y = \left[-\frac{1}{\lambda}\log\left\{1+\left\{\frac{1}{\log\delta}\log\left(1-\frac{(\delta-1)\,u}{\delta}\right)\right\}\right\}\right]^{1/\beta}. \tag{14}$$

The DPTW distribution has a density plot that can take on a diversity of shapes, including symmetrical, or right-skewed and decreasing. Figure (1) (left) shows some examples of these shapes. The HRF on the other hand, can take on the shapes of a constant, increasing j and a reverse-j. Figure (1) (right) shows some examples of these shapes.

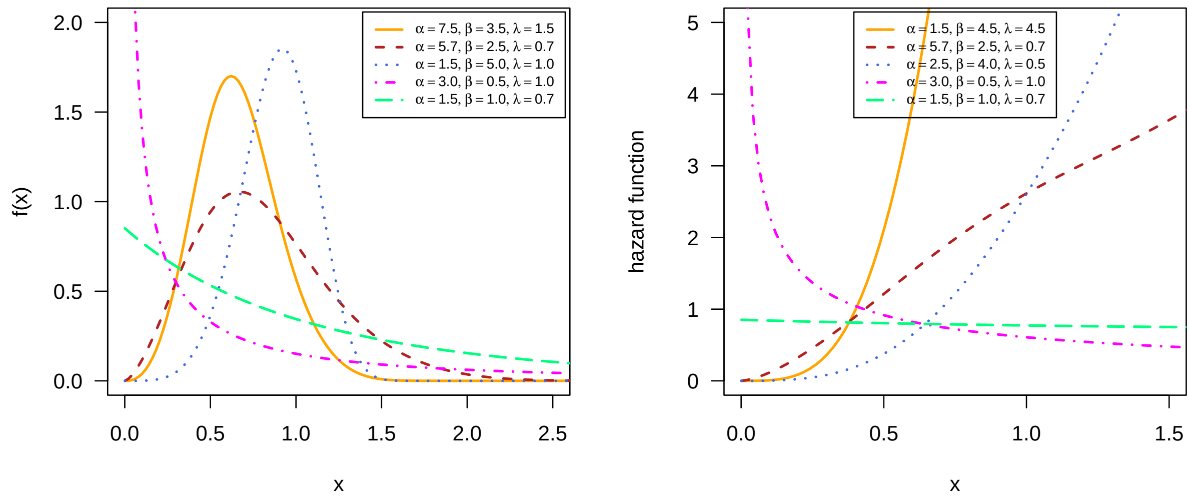

Figure 1: Shapes of PDF and HRF of DPTW distribution.

4 Statistical Properties

4.1 Linear form of DPTW distribution

Through some mathematical calculations, the PDF (10) of DPTW distribution can be expressed in linear form as

$$m(y;\delta,\beta,\lambda)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}y^{\beta-1}e^{-(j+1)\lambda y^{\beta}};\quad \delta>1, y>0, \tag{15}$$

where $\Delta_{ij}^{*}=\frac{\beta\lambda\delta}{\delta-1}\left(-1\right)^{j}\binom{i}{j}\Delta_{i}$.

4.2 Moments

The r^{th} moment of DPTW distribution is

$$E\left[Y^{r}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\int_{0}^{\infty}y^{\beta+r-1}e^{-(j+1)\lambda y^{\beta}}dy=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{r}{\beta}+1\right)}{\left\{(j+1)\lambda\right\}^{\frac{r}{\beta}+1}}. \tag{16}$$

Mean and Variance

$$E\left[Y\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{1}{\beta}+1\right)}{\left\{(j+1)\lambda\right\}^{\frac{1}{\beta}+1}}. \tag{17}$$

and

$$E\left[Y^{2}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{2}{\beta}+1\right)}{\left\{(j+1)\lambda\right\}^{\frac{2}{\beta}+1}}.$$

$$V[Y]=E\left[Y^{2}\right]-\left[E\left(Y\right)\right]^{2}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{2}{\beta}+1\right)}{\left\{(j+1)\lambda\right\}^{\frac{2}{\beta}+1}}-\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{1}{\beta}+1\right)}{\left\{(j+1)\lambda\right\}^{\frac{1}{\beta}+1}}\right]^{2}. \tag{18}$$

Using equations 17 and 18, we have computed the mean and variance of the DPTW distribution and results are displayed in Table 1.

Table 1: Mean and variance of DPTW for the various values of the parameters

δ, β and λ	$E(Y)$	$E(Y^2)$	$V(X)$
1.5, 0.5, and 0.75	2.5323	51.9262	45.5137
2.5, 1.5, and 1.25	0.7968	5.5693	4.9343
5.0, 1.5, and 0.5	0.4516	2.0151	1.8111
7.5, 2.5, and 1.75	0.2547	0.7903	0.7254

4.3 Moment generating function

For any real number t, the MGF of DPTW distribution can be defined as

$$D_Y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}\Delta_{ij}^{*}\int_0^{\infty} y^{\beta+k-1}e^{-(j+1)\lambda y^{\beta}}\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}\Delta_{ij}^{*}\frac{\beta^{-1}\Gamma\left(\frac{k}{\beta}+1\right)}{\{(j+1)\,\lambda\}^{\frac{k}{\beta}+1}}.$$

4.4 Order statistics

Let $y_i(i=1,...,n) \sim DPTW(y_i;\delta,\beta,\lambda)$ with CDF $M(y_i;\delta,\beta,\lambda)$ and PDF $m(y_i;\delta,\beta,\lambda)$. If $m_r(y)$ denotes the PDF of r^{th} order statistic $Y_{(r)}$, then their CDF and PDF are given by

$$M_r(y) = I_{M(y)}(r, n-r+1).$$

$$m_r(y) = \frac{d}{dy}\left[M_r(y)\right] = \frac{d}{dy}\left[I_{M(y)}(r,n-r+1)\right] = \frac{1}{B(r,n-r+1)}M^{r-1}(y)m(y)\left[1-M(y)\right]^{n-r}.$$

$$m_r(y) = \frac{1}{B(r,n-r+1)}\frac{\delta\beta\lambda(\log\delta)}{\delta-1}\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}y^{\beta-1}e^{-\lambda y^{\beta}}\left[Z_y\left(\delta,\beta,\lambda\right)\right]^{r-1}\left[1-Z_y\left(\delta,\beta,\lambda\right)\right]^{n-r},$$

where $Z_y\left(\delta,\beta,\lambda\right) = \frac{\delta}{\delta-1}\left(1-\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}\right)$, $I_M\left(a,b\right) = \int_0^t t^{a-1}\left(1-t\right)^{b-1}dt$ and $B\left(a,b\right)$ are incomplete and standard beta functions respectively. The CDF and PDF of first-order statistic $Y_{(1)}$ are given by

$$M_1(y) = 1-\left[1-Z_y\left(\delta,\beta,\lambda\right)\right]^{n}; y>0.$$

$$m_1(y) = \frac{n\delta\beta\lambda(\log\delta)}{\delta-1}\left[1-Z_y\left(\delta,\beta,\lambda\right)\right]^{n-1}\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}y^{\beta-1}e^{-\lambda y^{\beta}}; y>0.$$

The CDF and PDF of first-order statistic $Y_{(n)}$ are given by

$$M_n(y) = \left[Z_y\left(\delta,\beta,\lambda\right)\right]^{n}; y>0.$$

$$m_n(y) = \frac{n\delta\beta\lambda(\log\delta)}{\delta-1}\left[Z_y\left(\delta,\beta,\lambda\right)\right]^{n-1}\delta^{-\left(1-e^{-\lambda y^{\beta}}\right)}y^{\beta-1}e^{-\lambda y^{\beta}}; y>0.$$

The joint PDF of r^{th} and s^{th} order statistics are given by

$$m_{rs}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}M^{r-1}(x).m(x)\left[M(y)-M(x)\right]^{s-r-1}m(y).\left[1-M(y)\right]^{n-s}.$$

$$m_{rs}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left(\delta^{-\left(2-e^{-\lambda y^{\beta}}-e^{-\lambda x^{\beta}}\right)}(xy)^{\beta-1}e^{-\lambda\left(y^{\beta}+x^{\beta}\right)}\right)\left[\frac{\delta\beta\lambda(\log\delta)}{\delta-1}\right]^{2}$$
$$\left[Z_x\left(\delta,\beta,\lambda\right)\right]^{r-1}\left[Z_y\left(\delta,\beta,\lambda\right)-Z_x\left(\delta,\beta,\lambda\right)\right]^{s-r-1}\left[1-Z_y\left(\delta,\beta,\lambda\right)\right]^{n-s}; (x,y)\in R\times R.$$

The joint PDF of the 1^{st} and n^{th} order statistics is given by

$$m_{1n}(x,y) = n(n-1)\left[M(y)-M(x)\right]^{n-2} m(x).m(y)$$

$$m_{1n}(x,y) = n(n-1)\left(\frac{\delta\beta\lambda(\log\delta)}{\delta-1}\right)^2 \left[Z_y(\delta,\beta,\lambda) - Z_x(\delta,\beta,\lambda)\right]^{n-2} (xy)^{\beta-1} e^{-\lambda\left(y^\beta + x^\beta\right)} \delta^{-\left(2-e^{-\lambda x^\beta}-e^{-\lambda y^\beta}\right)},$$

where $Z_x(\delta,\beta,\lambda) = \frac{\delta}{\delta-1}\left(1-\delta^{-\left(1-e^{-\lambda x^\beta}\right)}\right)$.

5 Statistical Inference

5.1 Estimation

Let $y_i(i=1,...,n) \sim DPTW(y_i;\delta,\beta,\lambda)$ with PDF $m(y_i;\delta,\beta,\lambda)$, then the log-likelihood function can be calculated as

$$l(\underline{y};\delta,\beta,\lambda) = n\log(\delta\beta\lambda) + n\log(\log\delta) - n\log(\delta-1) - \log\delta\sum_{i=1}^{n}\left(1-e^{-\lambda y_i^\beta}\right) + (\beta-1)\sum_{i=1}^{n}\log y_i - \lambda\sum_{i=1}^{\infty} y_i^\beta. \quad (19)$$

Differentiating equation (19) with respect to associated parameters, we get

$$\frac{\partial l}{\partial\delta} = \frac{n}{\delta\log\delta} - \frac{n}{\delta-1} + \frac{1}{\delta}\sum_{i=1}^{n} e^{-\lambda y_i^\beta}.$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \lambda\log\delta\sum_{i=1}^{n} e^{-\lambda y_i^\beta} y_i^\beta \log y_i + \sum_{i=1}^{n}\log y_i - \lambda\sum_{i=1}^{\infty} y_i^\beta \log y_i.$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} + \log\delta\sum_{i=1}^{n} e^{-\lambda y_i^\beta} y_i^\beta - \sum_{i=1}^{\infty} y_i^\beta.$$

By solving the above three non-linear equations using suitable software one can obtain the estimates under the maximum likelihood estimation (MLE) method.

5.2 Cramer-Rao (CR) inequality

If $T(y_1,...,y_n)$ is an unbiased estimator for $g(\psi)$, a function of parameter ψ, then

$$Var[T(y_1,...,y_n)] \geqslant \frac{\left\{\frac{d}{d\psi}g(\psi)\right\}^2}{E\left(\frac{\partial}{\partial\psi}logL\right)^2} = \frac{\{g'(\psi)\}^2}{I(\psi)},$$

where $I(\psi)$ is the information on ψ, supplied by the sample. To define CR lower bound (CRLB) for δ when β and λ are known. The CRLB for an unbiased estimator $T_1(y_1,...,y_n)$ of a parameter δ is given by $\frac{1}{I(\delta)}$, where

$$I(\delta) = -E\left[\frac{\partial^2 l}{\partial\delta^2}\right] = n\left(\delta^{-2}(\log\delta)^{-2} + \delta^{-2}(\log\delta)^{-1}\right) + n(\delta-1)^{-2} + \delta^{-2}\sum_{i=1}^{n} E\left(e^{-\lambda y_i^\beta}\right)$$

and

$$\frac{\partial^2 l}{\partial\delta^2} = -n\left(\delta^{-2}(\log\delta)^{-2} + \delta^{-2}(\log\delta)^{-1}\right) - n(\delta-1)^{-2} - \delta^{-2}\sum_{i=1}^{n} e^{-\lambda y_i^\beta}.$$

Again CRLB for β when δ and λ are known, then CRLB for an unbiased estimator $T_2(y_1, ..., y_n)$ of a parameter β is given by $\frac{1}{I(\beta)}$, where

$$I(\beta) = -E\left[\frac{\partial^2 l}{\partial \beta^2}\right]$$
$$= n\beta^{-2} + \lambda \log \delta \sum_{i=1}^{n} E\left\{\left(e^{-\lambda y_i^\beta} y_i^\beta \log y_i - \lambda e^{-\lambda y_i^\beta} y_i^{2\beta} \log y_i\right) \log y_i\right\} + \lambda \sum_{i=1}^{\infty} E\left\{y_i^\beta (\log y_i)^2\right\}$$

and

$$\frac{\partial^2 l}{\partial \beta^2} = -n\beta^{-2} - \lambda \log \delta \sum_{i=1}^{n} \left(e^{-\lambda y_i^\beta} y_i^\beta \log y_i - \lambda e^{-\lambda y_i^\beta} y_i^{2\beta} \log y_i\right) \log y_i - \lambda \sum_{i=1}^{\infty} y_i^\beta (\log y_i)^2.$$

CRLB for λ when δ and β are known, then for an unbiased estimator $T_3(y_1, ..., y_n)$ of a parameter λ is given by $\frac{1}{I(\lambda)}$, where

$$I(\lambda) = -E\left[\frac{\partial^2 l}{\partial \lambda^2}\right] = n\lambda^{-2} + \lambda \log \delta \sum_{i=1}^{n} E\left\{e^{-\lambda y_i^\beta} y_i^{2\beta}\right\}$$

and

$$\frac{\partial^2 l}{\partial \lambda^2} = -n\lambda^{-2} - \lambda \log \delta \sum_{i=1}^{n} \left\{y_i^{2\beta} e^{-\lambda y_i^\beta}\right\}.$$

5.3 Asymptotical properties

As n approaches infinity, a steady solution to the likelihood equation tends towards a normal distribution centered around the true value θ_0. Consequently, $\hat{\theta}$ asymptotically follows a normal distribution with mean θ_0 and variance $\frac{1}{I(\theta_0)}$, expressed as $\hat{\theta} \sim N\left(\theta_0, \frac{1}{I(\theta_0)}\right)$. Particularly $\hat{\delta}, \hat{\beta}$ and $\hat{\lambda}$ are distributed asymptotically $N\left(\delta, \frac{1}{I(\delta)}\right)$, $N\left(\beta, \frac{1}{I(\beta)}\right)$ and $N\left(\lambda, \frac{1}{I(\lambda)}\right)$ respectively as $n \to \infty$.

5.4 Pivotal quantity (PQ)

Let $y_i(i = 1, 2, ..., n) \sim DPTW(y; \delta, \beta, \lambda)$ with CDF $M(y_i; \delta, \beta, \lambda)$, then pivotal quantity is defined as

$$-2\sum_{i=1}^{n} \ln\left[M(y_i; \delta, \beta, \lambda)\right] \sim \chi^2_{2n} \quad and \ -2\sum_{i=1}^{n} \ln\left[1 - M(y_i; \delta, \beta, \lambda)\right] \sim \chi^2_{2n}.$$

$$PQ = -2\sum_{i=1}^{n} \ln\left[Z_{y_i}(\delta, \beta, \lambda)\right] \sim \chi^2_{2n} \quad and \ PQ = -2\sum_{i=1}^{n} \ln\left[1 - Z_{y_i}(\delta, \beta, \lambda)\right] \sim \chi^2_{2n},$$

where χ^2_{2n} represent the chi-square distribution with $2n$ degree of freedom. Let $x_i(i = 1, 2, ..., m) \sim DPTW(x; \delta, \beta, \lambda)$ and $y_i(i = 1, 2, ..., n) \sim DPTW(y; \delta, \beta, \lambda)$ are two independent random variables with CDF $M(x_i; \delta, \beta, \lambda)$ and $M(y_i; \delta, \beta, \lambda)$ respectively, then $\frac{PQ_1}{PQ_2} \sim Beta_2(m, n)$ and $\frac{PQ_1}{PQ_1 + PQ_2} \sim Beta_1(m, n)$ and $\frac{n}{m}\frac{PQ_1}{PQ_2} \sim F(m, n)$, where

$$PQ_1 = -2\sum_{i=1}^{n} \ln\left[\frac{\delta}{\delta - 1}\left(1 - \delta^{-\left(1 - e^{-\lambda x_i^\beta}\right)}\right)\right]$$

and

$$PQ_2 = -2\sum_{i=1}^{n} \ln\left[\frac{\delta}{\delta - 1}\left(1 - \delta^{-\left(1 - e^{-\lambda y_i^\beta}\right)}\right)\right].$$

Generally, PQ is used to construct CI for a single parametric model. For this model, we used PQ to construct CI for a small sample for one parameter keeping others constant.

5.5 Confidence interval for large sample

Subject to regularity conditions, the asymptotic normality of the first derivative of the logarithm of the likelihood function with respect to parameter θ, denoted as $\frac{\partial \log L}{\partial \theta}$, is characterized by a mean of zero and a variance as follows

$$Var\left(\frac{\partial \log L}{\partial \theta}\right) = E\left(\frac{\partial \log L}{\partial \theta}\right)^2 = -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right)$$

Hence for large n,

$$T = \frac{\frac{\partial \log L}{\partial \theta}}{\sqrt{Var\left(\frac{\partial \log L}{\partial \theta}\right)}} \sim N(0,1)$$

The results enable us to obtain a confidence interval for the parameter θ in a large sample. Thus for a large sample, the confidence interval for θ with confidence coefficient $(1-b)\%$ is obtained by converting the inequalities in $P(|Z| \leqslant \gamma_b) = 1-b$, where γ_b is given by

$$\frac{1}{(2\pi)^{1/2}} \int_{-\gamma_b}^{\gamma_b} \exp\left(-t^2/2\right) dt = 1-b.$$

Thus Confidence interval for δ, β and λ are given by $\widehat{\delta} \pm \gamma_{b/2} SE\left(\widehat{\delta}\right)$, $\widehat{\beta} \pm \gamma_{b/2} SE\left(\widehat{\beta}\right)$ and $\widehat{\lambda} \pm \gamma_{b/2} SE\left(\widehat{\lambda}\right)$ at the confidence coefficient $(1-b)\%$.

6 Simulation Study

We utilized the maxLik R package, which was developed by Henningsen and Toomet [5], to generate samples from the quantile function specified in the equation (13) for various parameter combinations of the DPTW distribution. For each sample, we then computed the MLEs using the `maxLik()` function and the BFGS algorithm. This allowed us to investigate parameter estimation issues and determine the direction and size of bias (i.e., overestimation or underestimation) of the MLEs. In our simulation, we employed sample sizes ranging from 120 to 370 in increments of 50, and we repeated the process 1000 times to obtain estimates of bias and mean square error (MSE). We presented the results in the Tables 2, 3, and 4, which report the bias and MSEs for MLEs of each parameter. Our findings indicated that for three different parameter combinations, the bias and MSE decreased as the sample size increased. This suggests that the MLE method is asymptotically efficient, consistent, and follows the invariance property. However, one of the parameters, delta, in the proposed model exhibited slightly high bias values in the simulation study. We examined the bias of this parameter for large sample sizes and found that the bias values were small. Since the biases of the remaining parameters were nearly zero, we decided not to include all these results in the manuscript. Additionally, delta's contribution to the shape of the distribution is minimal, so the goodness of fit of the model remains unaffected.

Table 2: Simulation results for MLEs ($\delta = 5.0, \beta = 0.5, \lambda = 1.25$)

n	Bias			MSE		
	δ	β	λ	δ	β	λ
120	6.9424	-0.0209	0.2346	614.3530	0.0046	0.4713
170	6.8734	-0.0123	0.1413	563.0340	0.0032	0.3278
220	5.9442	-0.0099	0.0942	443.2450	0.0022	0.2409
270	5.5046	-0.0088	0.0895	408.8020	0.0022	0.2264
320	5.5526	-0.0073	0.0684	423.3870	0.0017	0.1955
370	4.2720	-0.0087	0.0833	287.1260	0.0017	0.1963

Table 3: Simulation results for MLEs ($\delta = 5.5, \beta = 0.75, \lambda = 1.5$)

n	Bias			MSE		
	δ	β	λ	δ	β	λ
120	8.0465	-0.0224	0.2160	697.1780	0.0093	0.5421
170	6.2254	-0.0241	0.1596	439.6600	0.0074	0.4160
220	6.9088	-0.0229	0.1491	771.6680	0.0062	0.3815
270	6.0974	-0.0156	0.1079	484.9720	0.0046	0.3093
320	6.3304	-0.0116	0.0810	479.7390	0.0041	0.2798
370	6.0813	-0.0112	0.0756	578.0470	0.0035	0.2405

Table 4: Simulation results for MLEs ($\delta = 2.5, \beta = 0.25, \lambda = 0.75$)

n	Bias			MSE		
	δ	β	λ	δ	β	λ
120	5.3011	-0.0066	0.1186	419.2490	0.0013	0.1945
170	4.9076	-0.0038	0.0657	320.4290	8.00E-04	0.1269
220	3.6058	-0.0053	0.0721	210.5540	8.00E-04	0.1248
270	3.0040	-0.0049	0.0627	157.3500	7.00E-04	0.1096
320	2.2869	-0.0024	0.0425	104.0350	6.00E-04	0.0892
370	2.6803	-0.0031	0.0522	176.5600	6.00E-04	0.0912

7 Application

Using two real datasets, we demonstrate the application of the DPTW distribution in this section. The datasets used for the suggested distribution applications are presented below.

7.1 Dataset-I

In this section, we demonstrate how the DPTW model can be applied using a real dataset that has been previously utilized by other researchers. The dataset we utilized comprises 100 observations on the breaking stress of carbon fibers, measured in gigabars (Gba). The datasets can be found in Nichols and Padgett [21] and Sapkota and Kumar [24].
3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

7.2 Dataset-II

The second actual set of data includes information on the duration of remission (in months) for a group of 128 individuals who have been diagnosed with bladder cancer. This data was collected from a random sample of patients and was originally published by Lee and Wang [11]. It can also be found in Sapkota [23].
0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

For both data sets, we have displayed the boxplots and total time on the test (TTT) plots in Figures 2 and 3 and observed that the data set I have almost symmetrical and increasing hazard rate whereas data set II has right skewed and bathtub shaped hazard rate. We have used the R package `AdequacyModel` developed by [1] to generate TTT plots.

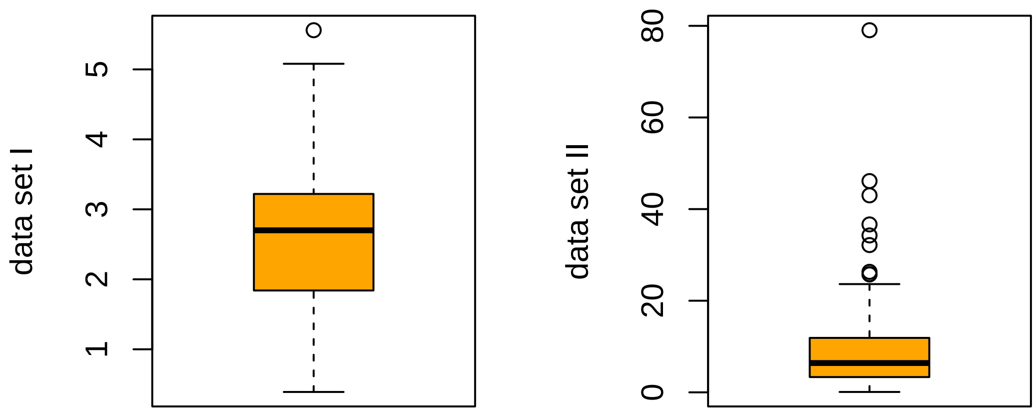

Figure 2: Boxplots of the dataset I and II.

7.3 Model analysis

We have computed several well-known goodness-of-fit statistics to analyze both data sets I and II. The fitted models were evaluated using various metrics, including the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogrov-Smirnov (KS), and Cramer-von Mises (CVM) with corresponding p-values. All essential computations were performed using the R software, for more detail see [17, 22]. To compare the fitting capability of the DPTW model, we have selected several models such as the inverse Weibull (IWeib), Weibull, alpha power inverse Pareto (APIP) [8], alpha power Weibull quantile (APWQ) [19], APT-Weibull (APTW) [20], and new APT-Weibull (NAPTW) [3]. We have presented KS plots and Quantile-Quantile (QQ) plots for both datasets in the Figures (4) and (5). Our analysis indicates that the suggested model can effectively fit the real datasets. The estimated values of the parameters (Par) and their associated standard errors (SE) for both datasets were presented in the Tables 5 and 6, which were obtained using the MLE method. Additionally, the Tables 7 and 8 showcase model selection criteria such as log-likelihood, HQIC, and AIC, and goodness of fit statistics such as KS, AD, and CVM for both datasets. Our observations show that the GOIChE model has the least statistics compared to the IWeib, Weibull, IWeib, APIP, APWQ, APTW, and NAPTW distributions, along with the corresponding highest p-values. For the dataset-II, the suggested model performed as similar as performed by NAPTW model. This indicates that the DPTW model is more flexible and provides a good fit. Furthermore, we have provided graphical illustrations of the fitted models in Figures (6) and (7), which support our findings that the DPTW model outperforms the other candidate models.

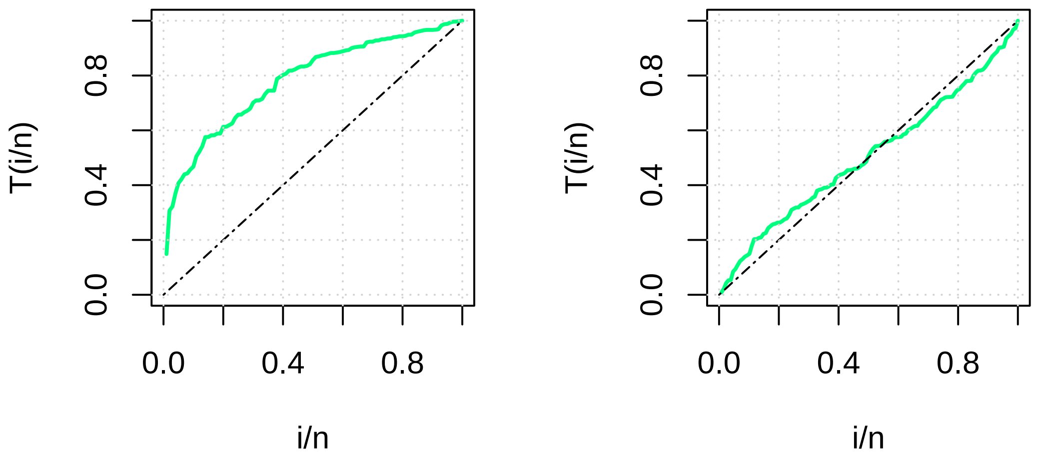

Figure 3: TTT plots of dataset I and II .

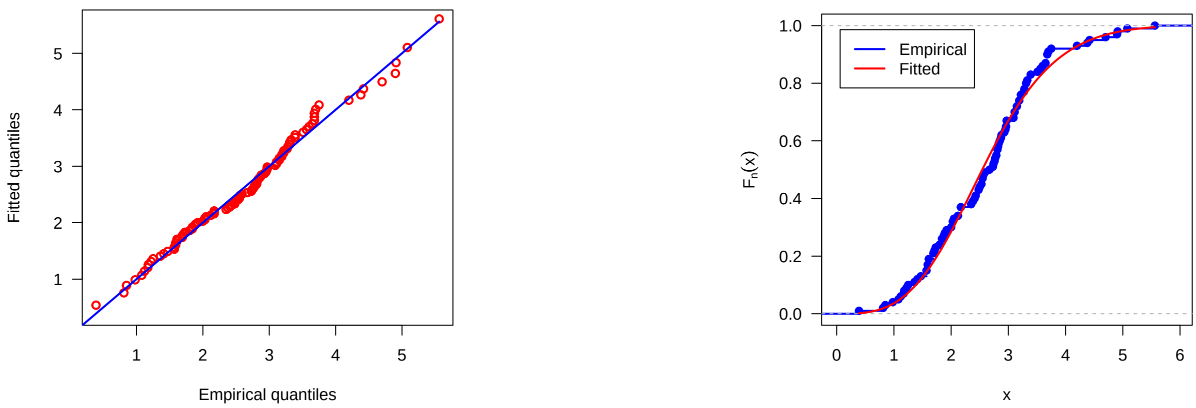

Figure 4: QQ and KS plots of DPTW distribution (dataset-I).

Table 5: MLEs with SE (dataset-I)

Model	**Par**	**SE**	**Par**	**SE**	**Par**	**SE**
$DPTW(\delta, \beta, \lambda)$	9.1663	5.2699	3.2415	0.2539	0.0150	0.0058
$IWeib(\beta, \lambda)$	1.7690	0.1121	3.0882	0.3283	–	–
$Weibull(\beta, \lambda)$	0.0490	0.0139	2.7928	0.2140	–	–
$APIP(\beta, \theta)$	11.1245	3.7479	1.0645	0.0000	–	–
$APWQ(\alpha, \beta, \theta)$	1.0000	0.6835	2.4830	0.0582	0.0722	0.0234
$APTW(\beta, \lambda, \theta)$	9.0285	6.6132	2.1089	0.1036	0.1768	0.0488
$NAPTW(\alpha, \beta, \lambda)$	10.4475	5.4553	1.7934	0.0714	0.2965	0.0353

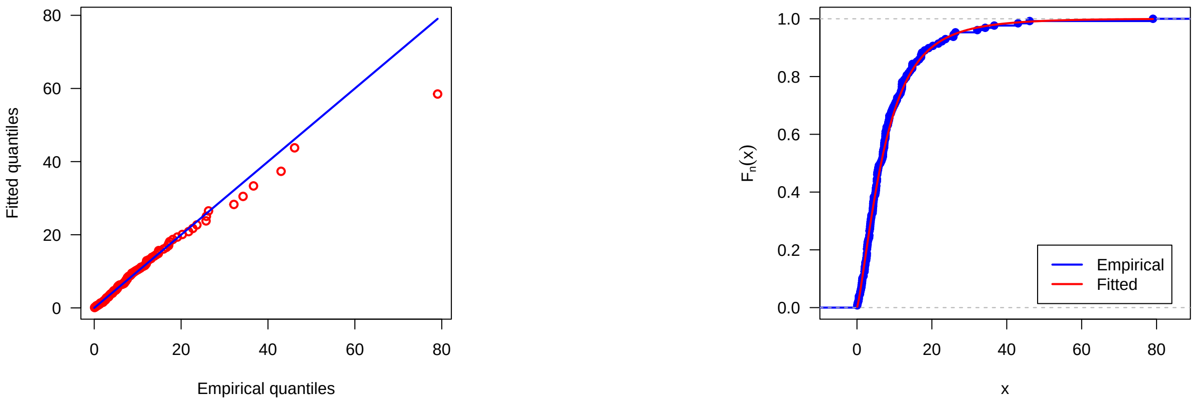

Figure 5: QQ and KS plots of DPTW distribution (dataset-II).

Table 6: MLEs with SE (dataset-II)

Model	**Par**	**SE**	**Par**	**SE**	**Par**	**SE**
$DPTW(\delta, \beta, \lambda)$	52.5472	4.2682	1.2702	0.0853	0.0175	0.0040
$IWeib(\beta, \lambda)$	0.7521	0.0424	2.4311	0.2196	–	–
$Weibull(\beta, \lambda)$	0.0939	0.0191	1.0478	0.0676	–	–
$APIP(\beta, \theta)$	77.8931	2.9822	1.1804	0.1247	–	–
$APWQ(\alpha, \beta, \theta)$	5.4624	3.4923	1.2776	0.1224	0.0315	0.0158
$APTW(\beta, \lambda, \theta)$	139.3390	74.6275	0.5946	0.0347	0.6511	0.0347
$NAPTW(\alpha, \beta, \lambda)$	93.5553	4.1734	0.5573	0.0383	0.7439	0.0733

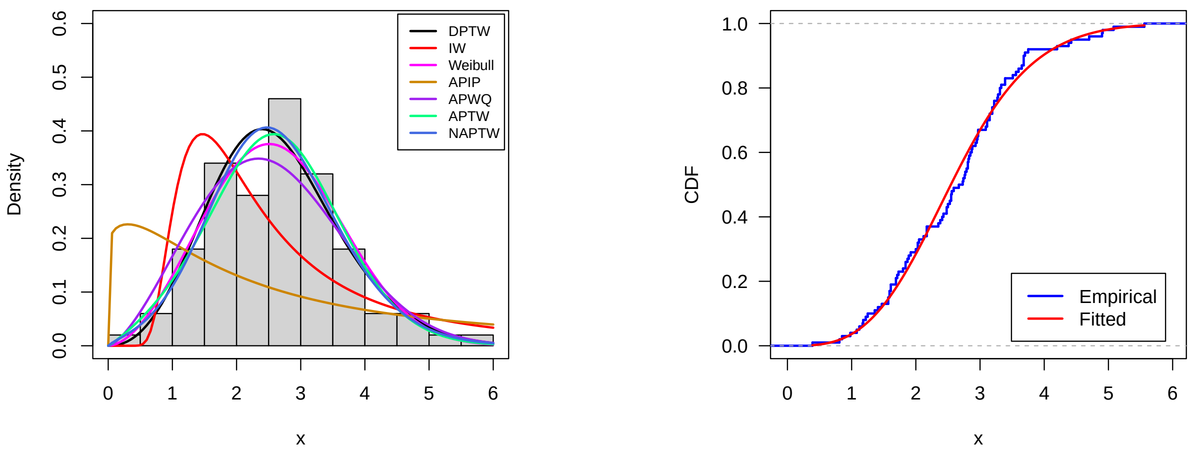

Figure 6: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-I).

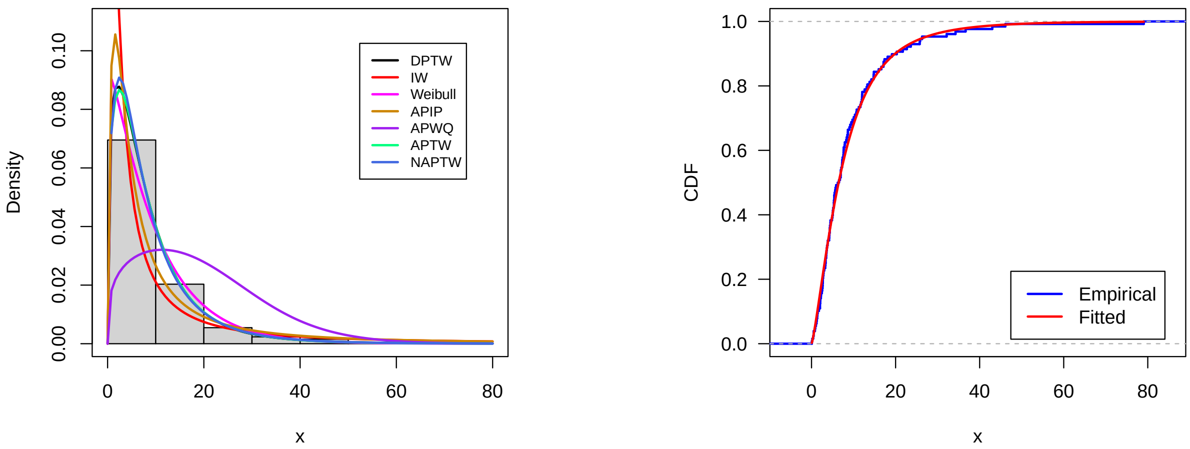

Figure 7: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-II).

Table 7: Some model selection and goodness-of-fit statistics (dataset-I)

Model	**-2logL**	**AIC**	**HQIC**	**KS**	**p(KS)**	**CVM**	**p(CVM)**	**AD**	**p(AD)**
DPTW	282.6701	288.6701	291.8332	0.0720	0.6770	0.0820	0.6813	0.4500	0.7977
IWeib	346.2879	350.2879	352.3966	0.1777	0.0036	0.8875	0.0044	5.3496	0.002
Weibull	283.0586	287.0586	289.1673	0.0605	0.8578	0.0633	0.7942	0.4177	0.8307
APIP	448.3450	452.3450	454.4537	0.3759	0.0000	3.9501	0.0000	19.978	0.0000
APWQ	285.2749	291.2749	294.4380	0.0876	0.4267	0.1679	0.3399	0.9582	0.3797
APTW	283.1754	289.1754	292.3385	0.0534	0.9379	0.0537	0.8544	0.3787	0.8692
NAPTW	282.9462	288.9462	292.1093	0.0649	0.7939	0.0676	0.7675	0.3957	0.8527

Table 8: Some model selection and goodness-of-fit statistics (dataset-II)

Model	**-2logL**	**AIC**	**HQIC**	**KS**	**p(KS)**	**CVM**	**p(CVM)**	**AD**	**p(AD)**
DPTW	820.4086	826.4086	829.8850	0.0461	0.9481	0.0384	0.9417	0.2487	0.9710
IWeib	888.0015	892.0015	894.3191	0.1408	0.0125	0.9787	0.0027	6.1183	9.00E-04
Weibull	828.1738	832.1738	834.4913	0.0700	0.5569	0.1537	0.3789	0.9577	0.3801
APIP	847.5875	851.5875	853.9051	0.1291	0.0280	0.5659	0.0270	3.7763	0.0113
APWQ	826.5404	832.5404	836.0167	0.0687	0.5818	0.1444	0.4074	0.8503	0.4459
APTW	820.8125	826.8125	830.2889	0.0440	0.9655	0.0366	0.9506	0.2476	0.9716
NAPTW	819.8662	825.8662	829.3426	0.0383	0.9920	0.0267	0.9861	0.1700	0.9965

8 Conclusion

In this investigation, a pioneering family of distributions named the delta power transformation (DPTW) family has been developed. Drawing inspiration from the ATP methodology, we selected the Weibull distribution as the fundamental framework for this novel family. The DPTW distribution showcases a spectrum of hazard function shapes, encompassing constant, monotonically increasing, J-shaped, and reverse-J-shaped patterns. Utilizing the MLE technique, we delved into the statistical properties of this distribution, estimating its parameters. A Monte Carlo simulation was conducted to assess the accuracy of our estimation method, revealing a decline in biases and mean square errors with increasing sample sizes, even with modest samples.

To showcase the practical utility of the DPTW distribution, we applied it to two real-world datasets and compared its performance against six established models using model selection criteria and goodness-of-fit tests. Our analysis demonstrated that the DPTW distribution consistently outperformed the other models for dataset-I. For dataset-II, the NAPTW model performed similarly to the DPTW model, highlighting its potential across various domains such as medical science, reliability engineering, and survival analysis. Additionally, the delta power transformation family of distributions holds promise for future model development, providing a versatile framework for tackling emerging challenges and advancing research in diverse fields.

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