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Inequalities for Means Regarding the Trigamma Function

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Abstract: *Let $\mathcal{G}(\alpha,\beta)$, $\mathcal{A}(\alpha,\beta)$ and $\mathcal{H}(\alpha,\beta)$, respectively, be the geometric mean, arithmetic mean and harmonic mean of α and β. In this paper, we prove that $\mathcal{G}(\psi'(z),\psi'(1/z)) \geq \pi^2/6$, $\mathcal{A}(\psi'(z),\psi'(1/z)) \geq \pi^2/6$ and $\mathcal{H}(\psi'(z),\psi'(1/z)) \leq \pi^2/6$. This extends the previous results of Alzer and Jameson regarding the digamma function ψ. The mathematical tools used to prove the results include convexity, concavity and monotonicity properties of certain functions as well as the convolution theorem for Laplace transforms.*

Keywords: Gamma function, Digamma function, Trigamma function, Harmonic mean inequality

1 Introduction

The classical gamma function which is an extension of the factorial function is frequently defined as

$$\Gamma(z) = \int_0^\infty r^{z-1} e^{-r} dr$$

for $z > 0$. Closely connected to the gamma function is the the digamma (or psi) function which is defined as

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = -\gamma + \int_0^\infty \frac{e^{-r}-e^{-zr}}{1-e^{-r}} dr, \tag{1}$$

$$= \int_0^\infty \left(\frac{e^{-r}}{r} - \frac{e^{-zr}}{1-e^{-r}} \right) dr, \tag{2}$$

$$= -\gamma - \frac{1}{z} + \sum_{n=1}^\infty \frac{z}{n(n+z)}, \tag{3}$$

where γ is the Euler-Mascheroni constant. Derivatives of the digamma function which are called polygamma functions are defined as

$$\psi^{(c)}(z) = (-1)^{c+1} \int_0^\infty \frac{r^c e^{-zr}}{1-e^{-r}} dr, \tag{4}$$

$$= (-1)^{c+1} \sum_{n=0}^\infty \frac{c!}{(n+z)^{c+1}}, \tag{5}$$

for $z > 0$ and $c \in \mathbb{N}$. The particular case $\psi'(z)$ is what is referred to as the trigamma function. Also, it is well known in the literature that the integral

$$\frac{c!}{z^{c+1}} = \int_0^\infty r^c e^{-zr}\, dr \tag{6}$$

holds for $z > 0$ and $c \in \mathbb{N}_0$.

In 1974, Gautschi [11] presented an elegant inequality involving the gamma function. Precisely, he proved that, for $z > 0$, the harmonic mean of $\Gamma(z)$ and $\Gamma(1/z)$ is at least 1. That is,

$$\frac{2\Gamma(z)\Gamma(1/z)}{\Gamma(z)+\Gamma(1/z)} \geq 1, \tag{7}$$

for $z > 0$ and with equality when $z = 1$. As a direct consequence of (7), the inequalities

$$\Gamma(z) + \Gamma(1/z) \geq 2 \tag{8}$$

and

$$\Gamma(z)\Gamma(1/z) \geq 1 \tag{9}$$

are obtained for $z > 0$. Attributing to the importance of this inequality, some refinements and extensions have been investigated [1, 2, 3, 4, 5, 6, 12, 13].

In 2017, Alzer and Jameson [8] established a striking companion of (7) which involves the digamma function $\psi(z)$. They established that the inequality

$$\frac{2\psi(z)\psi(1/z)}{\psi(z) + \psi(1/z)} \geq -\gamma \tag{10}$$

holds for $z > 0$ and with equality when $z = 1$. Thereafter, Alzer [7] refined (10) by proving that

$$\frac{2\psi(z)\psi(1/z)}{\psi(z) + \psi(1/z)} \geq -\gamma \frac{2z}{z^2 + 1} \tag{11}$$

holds for $z > 0$ and with equality when $z = 1$.

In 2018, Yin et al. [25] extended inequality (10) to the k-analogue of the digamma function by establishing that

$$\frac{2\psi_k(z)\psi_k(1/z)}{\psi_k(z) + \psi_k(1/z)} \geq \frac{\ln^2 k + \gamma^2 - 2(\gamma + 1)\ln k}{k\left[\ln k + \psi(1/k)\right]} \tag{12}$$

for $z > 0$ and $\frac{1}{\sqrt[3]{3}} \leq k \leq 1$.

In 2020, Yildirim [24] improved on the inequality (12) by establishing that

$$\frac{2\psi_k(z)\psi_k(1/z)}{\psi_k(z) + \psi_k(1/z)} \geq \psi_k(1) \tag{13}$$

for $z > 0$ and $k > 0$. When $k = 1$, inequalities (12) and (13) both return to inequality (10).

In 2021, Bouali [10] extended inequalities (7) and (10) to the q-analogues of the gamma and digamma functions by proving that

$$\frac{2\Gamma_q(z)\Gamma_q(1/z)}{\Gamma_q(z) + \Gamma_q(1/z)} \geq 1 \tag{14}$$

for $z > 0$ and

$$\frac{2\psi_q(z)\psi_q(1/z)}{\psi_q(z) + \psi_q(1/z)} \geq \psi_q(1) \tag{15}$$

for $z > 0$ and $q \in (0, p_0)$, where $p_0 \simeq 3.239945$.

For similar results involving other special functions, one may refer to the works [14, 15, 16, 17, 18, 19, 20]. In the present investigation, our objective is to extend the results of Alzer and Jameson [8] to the trigamma function ψ' among other things. Specifically, we prove that

(a) for $z > 0$, the geometric mean of $\psi'(z)$ and $\psi'(1/z)$ can never be less than $\pi^2/6$.

(b) for $z > 0$, the arithmetic mean of $\psi'(z)$ and $\psi'(1/z)$ can never be less than $\pi^2/6$.

(c) for $z > 0$, the harmonic mean of $\psi'(z)$ and $\psi'(1/z)$ can never be greater than $\pi^2/6$.

We present our results in Section 2. In order to establish our results, we require the following preliminary definitions and lemmas.

Definition 1.1 ([21]). A function $H : \mathcal{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is referred to as GG-convex if

$$H(x^{1-k}y^k) \leq H(x)^{1-k}H(y)^k \tag{16}$$

for all $x, y \in \mathcal{I}$ and $k \in [0, 1]$. If the inequality in (16) is reversed, then H is said to be GG-concave.

Definition 1.2 ([21])**.** A function $H : \mathcal{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is referred to as GA-convex if

$$H(x^{1-k}y^k) \leq (1-k)H(x) + kH(y) \tag{17}$$

for all $x, y \in \mathcal{I}$ and $k \in [0,1]$. If the inequality in (17) is reversed, then H is said to be GA-concave.

Lemma 1.3 ([21])**.** *A function $H : \mathcal{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is GG-convex (or GG-concave) if and only if $\frac{zH'(z)}{H(z)}$ is increasing (or decreasing) on $\mathcal{I}$ respectively.*

Lemma 1.4 ([26])**.** *A function $H : \mathcal{I} \subseteq \mathbb{R}^+ \to \mathbb{R}$ is GA-convex if and only if*

$$H'(z) + zH''(z) \geq 0 \tag{18}$$

for all $z \in \mathcal{I}$. The function H is said to be GA-concave if and only if the inequality in (18) *is reversed.*

The following lemma is well known in the literature as the convolution theorem for Laplace transforms.

Lemma 1.5 ([23])**.** *Let $f(r)$ and $g(r)$ be any two functions with convolution $f * g = \int_0^r f(r-s)g(s)\,ds$. Then the Laplace transform of the convolution is given as*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}\,.$$

In other words,

$$\int_0^\infty \left[\int_0^r f(r-s)g(s)\,ds\right] e^{-zr}\,dr = \int_0^\infty f(r)e^{-zr}\,dr \int_0^\infty g(r)e^{-zr}\,dr. \tag{19}$$

Lemma 1.6 ([22])**.** *Let $-\infty \leq u < v \leq \infty$ and p and q be continuous functions that are differentiable on (u,v), with $p(u+) = q(u+) = 0$ or $p(v-) = q(v-) = 0$. Suppose that $q(z)$ and $q'(z)$ are nonzero for all $z \in (u,v)$. If $\frac{p'(z)}{q'(z)}$ is increasing (or decreasing) on (u,v), then $\frac{p(x)}{q(x)}$ is also increasing (or decreasing) on (u,v).*

2 Results

Theorem 2.1. *The function $\psi'(z)$ is GG-convex on $(0,\infty)$. In other words,*

$$\psi'(x^{1-k}y^k) \leq [\psi'(x)]^{1-k}\,[\psi'(y)]^k \tag{20}$$

is satisfied for $x > 0$, $y > 0$ and $k \in [0,1]$.

Proof. As a result of Lemma 1.3, it suffices to show that the function $z\frac{\psi''(z)}{\psi'(z)}$ is increasing on $(0,\infty)$ and this follows from Lemma 2 of [7]. □

Corollary 2.2. *The inequality*

$$\psi'(z)\psi'(1/z) \geq \left(\frac{\pi^2}{6}\right)^2 \tag{21}$$

holds for $z \in (0,\infty)$ and with equality when $z = 1$.

Proof. By letting $x = z$, $y = 1/z$ and $k = \frac{1}{2}$ in (20), we obtain

$$\sqrt{\psi'(z)\psi'(1/z)} \geq \psi'(1) = \frac{\pi^2}{6}$$

which gives the desired result. □

Lemma 2.3. *For $r > 0$, we have*

$$0 < \frac{re^{-r}}{1-e^{-r}} < 1. \tag{22}$$

Proof. By direct computation, we obtain

$$\mathcal{B}(r) = \frac{re^{-r}}{1-e^{-r}} = \frac{p_1(r)}{q_1(r)}$$

where $p_1(r) = re^{-r}$, $q_1(r) = 1 - e^{-r}$ and $p_1(0+) = q_1(0+) = 0$. Then

$$\frac{p_1'(r)}{q_1'(r)} = 1 - r$$

and then

$$\left(\frac{p_1'(r)}{q_1'(r)}\right)' = -1 < 0.$$

Thus, $\frac{p_1'(r)}{q_1'(r)}$ is decreasing and as a result of Lemma 1.6, the function $\mathcal{B}(r)$ is also decreasing. Hence

$$0 = \lim_{r\to\infty} \mathcal{B}(r) < \mathcal{B}(r) < \lim_{r\to 0+} \mathcal{B}(r) = 1$$

which completes the proof. $\square$

Theorem 2.4. *The function $\psi'(z)$ is GA-convex on $(0,\infty)$. In other words,*

$$\psi'(x^{1-k}y^k) \leq (1-k)\psi'(x) + k\psi'(y) \tag{23}$$

is satisfied for $x > 0$, $y > 0$ and $k \in [0,1]$.

Proof. As a result of Lemma 1.4, it suffices to show that

$$\phi(z) = \psi''(z) + z\psi'''(z) \geq 0 \tag{24}$$

for $z \in (0,\infty)$. By applying (4), (6) and Lemma 1.5, we obtain

$$\begin{aligned}
\frac{\phi(z)}{z} &= \frac{1}{z}\psi''(z) + \psi'''(z) \\
&= -\int_0^\infty e^{-zr}dr \int_0^\infty \frac{r^2e^{-zr}}{1-e^{-r}}dr + \int_0^\infty \frac{r^3e^{-zr}}{1-e^{-r}}dr \\
&= -\int_0^\infty \left[\int_0^r \frac{s^2}{1-e^{-s}}ds\right] e^{-zr}dr + \int_0^\infty \frac{r^3e^{-zr}}{1-e^{-r}}dr \\
&= \int_0^\infty \mathcal{A}(r)e^{-zr}dr
\end{aligned}$$

where

$$\mathcal{A}(r) = \frac{r^3}{1-e^{-r}} - \int_0^r \frac{s^2}{1-e^{-s}}ds.$$

Then by direct computations and as a result of (22), we have

$$\begin{aligned}
\mathcal{A}'(r) &= \frac{3r^2}{1-e^{-r}} - \frac{r^3e^{-r}}{(1-e^{-r})^2} - \frac{r^2}{1-e^{-r}} \\
&= \frac{r^2}{1-e^{-r}}\left[2 - \frac{re^{-r}}{1-e^{-r}}\right] \geq 0.
\end{aligned}$$

Hence $\mathcal{A}(r)$ is increasing and this implies that

$$\mathcal{A}(r) \geq \lim_{r\to 0+} \mathcal{A}(r) = 0.$$

Therefore, $\phi(z) \geq 0$ which completes the proof. $\square$

Remark 2.5. Inequality (24) implies that the function $z\psi''(z)$ is increasing.

Corollary 2.6. *The inequality*

$$\psi'(z) + \psi'(1/z) \geq \frac{\pi^2}{3} \tag{25}$$

holds for $z \in (0, \infty)$ and with equality when $z = 1$.

Proof. By letting $x = z$, $y = 1/z$ and $k = \frac{1}{2}$ in (23), we obtain

$$\frac{\psi'(z)}{2} + \frac{\psi'(1/z)}{2} \geq \psi'(1) = \frac{\pi^2}{6}$$

which gives the desired result. □

Lemma 2.7 ([9])**.** *For $z > 0$, the inequality*

$$\psi'(z)\psi'''(z) - 2\left[\psi''(z)\right]^2 \leq 0 \tag{26}$$

is satisfied.

Lemma 2.8. *For $z > 0$, the function*

$$F(z) = \frac{z\psi''(z)}{[\psi'(z)]^2} \tag{27}$$

is decreasing.

Proof. By applying Lemma 2.7, we obtain

$$\begin{aligned}
[\psi'(z)]^3 F'(z) &= \psi'(z)\psi''(z) + z\psi'(z)\psi'''(z) - 2z[\psi''(z)]^2 \\
&= \psi'(z)\psi''(z) + z\left[\psi'(z)\psi'''(z) - 2\left[\psi''(z)\right]^2\right] \\
&< 0.
\end{aligned}$$

Hence $F'(z) < 0$ which completes the proof. □

Theorem 2.9. *For $z > 0$, the inequality*

$$\frac{2\psi'(z)\psi'(1/z)}{\psi'(z) + \psi'(1/z)} \leq \frac{\pi^2}{6} \tag{28}$$

holds and equality is attained if $z = 1$.

Proof. The case for $z = 1$ is apparent. For this reason, we only prove the case for $z \in (0, 1) \cup (1, \infty)$. Let

$$\mathcal{K}(z) = \frac{2\psi'(z)\psi'(1/z)}{\psi'(z) + \psi'(1/z)} \quad \text{and} \quad \beta(z) = \ln \mathcal{K}(z)$$

for $z \in (0, 1) \cup (1, \infty)$. Then direct calculations gives

$$\beta'(z) = \frac{\psi''(z)}{\psi'(z)} - \frac{1}{z^2}\frac{\psi''(1/z)}{\psi'(1/z)} - \frac{\psi''(z) - \frac{1}{z^2}\psi''(1/z)}{\psi'(z) + \psi'(1/z)}$$

which implies that

$$z\left[\psi'(z) + \psi'(1/z)\right]\beta'(z) = z\frac{\psi''(z)}{\psi'(z)}\psi'(1/z) - \frac{1}{z}\frac{\psi''(1/z)}{\psi'(1/z)}\psi'(z).$$

This further gives rise to

$$\begin{aligned}
z\left[\frac{1}{\psi'(z)} + \frac{1}{\psi'(1/z)}\right]\beta'(z) &= z\frac{\psi''(z)}{[\psi'(z)]^2} - \frac{1}{z}\frac{\psi''(1/z)}{[\psi'(1/z)]^2} \\
&:= T(z).
\end{aligned}$$

As a result of Lemma 2.8, we conclude that $T(z) > 0$ if $z \in (0,1)$ and $T(z) < 0$ if $z \in (1,\infty)$. Thus, $\beta(z)$ is increasing on $(0,1)$ and decreasing on $(1,\infty)$. Accordingly, $\mathcal{K}(z)$ is increasing on $(0,1)$ and decreasing on $(1,\infty)$. Therefore, on both intervals, we have

$$\mathcal{K}(z) < \lim_{z\to 1} \mathcal{K}(z) = \psi'(1) = \frac{\pi^2}{6}$$

completing the proof. □

3 Conclusion

By using convexity, concavity and monotonicity properties of certain functions as well as the convolution theorem for Laplace transforms, we have proved that

(a) for $z > 0$, the geometric mean of $\psi'(z)$ and $\psi'(1/z)$ can never be less than $\pi^2/6$.

(b) for $z > 0$, the arithmetic mean of $\psi'(z)$ and $\psi'(1/z)$ can never be less than $\pi^2/6$.

(c) for $z > 0$, the harmonic mean of $\psi'(z)$ and $\psi'(1/z)$ can never be greater than $\pi^2/6$.

This extends the earlier results of Alzer and Jameson regarding the digamma function. In a future study, we will like to investigate whether it is possible to extend these results to the polygamma function.

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