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On Topological Structure of Total Paranormed Double Sequence Space $(\ell^2((X, ||.||), \bar{\gamma}, \bar{w}), G)$

Jagat Krishna Pokharel[1,*], Narayan Prasad Pahari[2], Gyan Prasad Paudel[3]

[1]Department of Mathematics Education, Sanothimi Campus, Tribhuvan University, Nepal
[2]Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal
[3]Central Campus of Science and Technology, Mid-West University, Surkhet, Nepal

*Correspondence to: Jagat Krishna Pokharel, Email: jagatpokhrel.tu@gmail.com

Abstract: *The aim of this paper is to introduce and study a new class $\ell^2((X, ||.||), \bar{\gamma}, \bar{\omega})$ of double sequences with their terms in a normed space X as a generalization of the familiar sequence space ℓ. Besides the investigation of the condition pertaining to the containment relations of the class $\ell^2((X, ||.||), \bar{\gamma}, \bar{\omega})$ of same kind in terms of $\bar{\gamma}$ and $\bar{\omega}$, our primary interest is to explore some of the preliminary results that characterize the linear topological structures of $\ell^2((X, ||.||), \bar{\gamma}, \bar{\omega})$ when topologized it with suitable natural paranorm.*

Keywords: Paranormed space, Sequence space, Double sequence

1 Introduction

We begin with recalling some notations and basic definitions that are used in this paper. The concept of paranorm is closely related to the linear metric space; see Wilansky [26] and its studies on sequence spaces were initiated by Maddox [9], and many others. Basarir and Altundag [1], Ghimire, & Pahari [2, 3], Parasar and Choudhary[15], Paudel and Pahari [18, 20, 21] and many others further studied the various types of paranormed sequence spaces.

A paranormed space (X, G) is a linear space X with the zero element θ together with the function $G : X \to \mathbf{R}^+$ (called a paranorm on X) which satisfy the following axioms

P.N1: G(θ)= 0 ;

P.N2: $G(x) = G(-x)$ for all $x \in X$;

P.N3: $G(x + y) \leq G(x) + G(y)$ for all $x, y \in X$; and

P.N4: Scalar multiplication is continuous

Note that the continuity of scalar multiplication is equivalent to
i. $G(x_k) \to 0$ and $\lambda_k \to \lambda$ as $k \to \infty$ then $G(\lambda_k, x_k) \to 0$ as $k \to \infty$, and
ii. $\lambda_k \to 0$ as $k \to \infty$ and x be any element in X, then $g(\lambda_k, x) \to 0$ see Wilansky [26].

A paranorm is called total if $G(x) = \theta$ implies $x = \theta$.

Let X be a normed space over the field of complex numbers. Let $\omega(X)$ denotes the space of all sequences $\bar{x} = (x_i)$ with $x_i \in X, i \geq 1$. We shall denote $\omega(\mathbb{C})$ by ω. Any subspace S of ω is then called a sequence space. A normed space valued sequence space or a generalized sequence space is a linear space of sequences with their terms in a normed space. Several workers like Gupta and Patterson [5], Kamthan and Gupta [6], Kolk [7], Köthe [8], Maddox [10], and Pahari [12] etc. have introduced and studied some properties of vector and scalar-valued single sequence spaces, when sequences are taken from a Banach space.

The concept of various types of linear spaces of single sequences and their special kind of convergence was studied by several workers for instances we refer a few: Pahari [12, 13, 14], Pokharel, Pahari and Ghimire [22], Srivastava and Pahari [24, 25]. They also studied the various types of topological structures of vector valued sequence spaces defined by Orlicz function endowed with suitable natural paranorms and extended some of them in 2-normed spaces. Recently, Paudel, Pahari and et al. [17, 18, 19, 20, 21, 22] has extended the the concepts of sequence space of complex numbers to the sequences of fuzzy real numbers.

The theory of single sequence spaces has also been extended to the spaces of double sequences and studied by several workers. Gupta and Kamthan [4], Morics [11] and many others have made significant contributions and enriched the theories in this direction. In the recent years, Savas [22], Subramanian et al. [25] and many others have introduced and studied various types of double sequence spaces using Orlicz function.

2 The Class $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ of Double Sequences

Let $\bar{w}=(w_{nk})$ and $\bar{v}=(v_{nk})$ be any double sequences of strictly increasing positive real numbers and $\bar{\gamma}=(\gamma_{nk})$ and $\bar{\mu}=(\mu_{nk}), n,k\geq 1$ be double sequences of non-zero complex numbers. Let $(X,||.||)$, and $(Y,||.||)$ be Banach spaces over the field of complex numbers and $B\left((X,||.||),Y\right)$ be the Banach space of all bounded linear operators from $(X,||.||)$ into Y. The zero element of the Banach spaces $X,Y,B\left((X,||.||),Y\right)$ will be denoted by θ.

Throughout the work, $\sum\sum$ will denote

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \quad \text{in the sense that} \qquad \lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}$$

We now introduce and study the following class of Banach space-valued double sequences

$$\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)=\left\{\bar{x}=(x_{nk}):x_{nk}\in X,n,k\geq 1 \text{ and } \lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}||\gamma_{nk}x_{nk}||^{w_{nk}}\to\theta \text{ as } n+k\to\infty\right\}$$

Further, when $\gamma_{nk}=1$ for all n and k, then $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ will be denoted by $\ell^2\left((X,||.||),\bar{w}\right)$ and when w_{nk} for all n and k; then $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ will be denoted by $\ell^2\left((X,||.||),\bar{\gamma}\right)$.
Further, by $\bar{w}=(w_{nk})\in\ell^2_{\infty}$, we mean $\sup w_{nk}<\infty$. We denote $A(\lambda)=\max(1,|\lambda|)$ and the zero element of this class by $\bar{\theta}=(\theta,\theta,\theta,\dots)$

3 Main Results

In this section, we investigate some conditions in terms of $\bar{w}$ and $\bar{\gamma}$ so that a class $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ is contained in or equal to another class of the same kind and then explore some of the preliminary results that characterize the linear topological structure of $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ when topologized it with suitable natural paranorm. As far as the linear space structure of $\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ over the field of complex numbers is concerned, we throughout take the coordinatewise operations, i.e, for

$$\bar{x}=(x_{nk}),\bar{y}=(y_{nk}) \text{ and scalar } \lambda,\ \bar{x}+\bar{y}=(x_{nk}+y_{nk}) \text{ and } \lambda\bar{x}=(\lambda x_{nk}).$$

Theorem 3.1. For any $\bar{w}$, $\ell^2\left((X,||.||),\bar{\mu},\bar{w}\right)\subset\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ if and only if

$$\limsup_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}<\infty.$$

Proof. Suppose $\limsup_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}<\infty$ and $\bar{x}\in\ell^2\left((X,||.||),\bar{\mu},\bar{w}\right)$. Then there exists a constant $L>0$ such that $|\gamma_{nk}|^{w_{nk}}<L|\mu_{nk}|^{w_{nk}}$ for all sufficiently large value of n,k. This means that for all sufficiently large values of n,k,

$$|\gamma_{nk}\ x_{nk}|^{w_{nk}}<L|\mu_{nk}\ x_{nk}|^{w_{nk}}$$

Thus $\sum\sum||\mu_{nk}x_{nk}||^{w_{nk}}<\infty$ implies that $\sum\sum||\gamma_{nk}x_{nk}||^{w_{nk}}<\infty$, i.e., $\bar{x}\in\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$ and hence $\ell^2\left((X,||.||),\bar{\mu},\bar{w}\right)\subset\ell^2\left((X,||.||),\bar{\gamma},\bar{w}\right)$.

Conversely, let the inclusion holds but $\limsup\limits_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}=\infty$. Then there exist subsequences $(n(i))$ of (n) and $(k(i))$ of (k) respectively such that for each $i\geq 1$,

$$|\gamma_{n(i)k(i)}|^{w_{n(i)k(i)}} > i|\mu_{n(i)k(i)}|^{w_{n(i)k(i)}} \tag{1}$$

Thus for $z_{nk}\in X$ with $||z_{nk}||=1$, the sequence $\bar{x}=(x_{nk})$ defined by

$$x_{nk}=\begin{cases} \mu_{nk}^{-1}i^{-2/w_{nk}}, n=n(i), k=k(i), i\geq 1 \\ \theta, \quad \text{otherwise} \end{cases} \tag{2}$$

is in $\ell^2\left((X,||.||),\bar{\mu},\bar{w}\right)$, since for $n=n(i),k=k(i),i\geq 1$ and in view of (1) and (2),

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\mu_{nk}x_{nk}\|^{w_{nk}}=\sum_{i=1}^{\infty}\left\|\mu_{n(i)k(i)}x_{n(i)k(i)}\right\|^{w_{n(i)k(i)}}=\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty.$$

But $\bar{x}$ does not belong to $\ell^2((X,\|.\|),\bar{\gamma},\bar{w})$, since for $n=n(i),k=k(i),i\geq 1$,

$$\begin{aligned}\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\mu_{nk}x_{nk}\|^{w_{nk}}&=\sum_{i=1}^{\infty}\left\|\mu_{n(i)k(i)}x_{n(i)k(i)}\right\|^{w_{n(i)k(i)}}\\&=\sum_{i=1}^{\infty}\left|\frac{\lambda_{n(i)k(i)}}{\mu_{n(i)k(i)}}\right|^{w_{n(i)k(i)}}\frac{1}{i^2}>\sum_{i=1}^{\infty}\frac{1}{i}=\infty,\end{aligned}$$

a contradiction. This completes the proof. □

Theorem 3.2. For any $\bar{w}=(w_{nk})$, $\ell^2((X,\|.\|),\bar{\gamma},\bar{w})\subset\ell^2((X,\|.\|),\bar{\mu},\bar{w})$

$$\text{if and only if } \liminf_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}>0.$$

Proof. Suppose $\liminf\limits_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}>0$ and $\bar{x}=(x_{nk})\in\ell^2((X,\|\|),.\bar{\gamma},\bar{w})$. Then there exists $I>0$ such that $I\left|\mu_{nk}\right|^{w_{nk}}<\left|\gamma_{nk}\right|^{w_{nk}}$ for all sufficiently large values of n,k. Thus

$$I\left\|\mu_{nk}x_{nk}\right\|^{w_{nk}}\leq\left\|\gamma_{nk}x_{nk}\right\|^{w_{nk}}$$

for all sufficiently large values of n,k. From the above inequality and we see that

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\mu_{nk}x_{nk}\|^{w_{nk}}<\infty$$

i.e., $\bar{x}\in\ell^2\left((X\|.\|),\bar{\mu},\bar{w}\right)$ and hence

$$\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})\subset\ell^2((X,\|\cdot\|),\bar{\mu},\bar{w}).$$

Conversely, let the inclusion holds but $\liminf\limits_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}=0$. Then there exist subsequences $(n(i))$ of (n) and $(k(i))$ of (k) respectively such that for each $i\geq 1$

$$i\left\|\gamma_{n(i)k(i)}\right\|^{w_{n(i)k(i)}}\geq\left\|\mu_{n(i)k(i)}\right\|^{w_{n(i)k(i)}} \tag{3}$$

For $z_{nk}\in X$ and $\|z_{nk}\|=1$, we define a sequence $\bar{x}=(x_{nk})$ by

$$x_{nk}=\begin{cases}\gamma_{nk}^{-1}i^{-2/w_{nk}}z_{nk}, k=k(i), i\geq 1\\ \theta,\ \text{otherwise.}\end{cases} \tag{4}$$

Then as proved in Theorem 3.1, for $n=n(i),k=k(i),i\geq 1$ and in view of (3) and (4), we can prove that $\bar{x}$ is in $\ell^2\left((X,\|.\|),\bar{\gamma},\bar{w}\right)$ and $\bar{x}\in\ell^2((X,\|\cdot\|),\bar{\mu},\bar{w})$, a contradiction. This completes the proof. □

On combining Theorems 3.1 and 3.2, we get

Theorem 3.3. For any $\bar{w}=(w_{nk})\,,\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})=\ell^2((X,\|\cdot\|),\bar{\mu},\bar{w})$ if and only if

$$0<\liminf_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}\leq\limsup_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}<\infty.$$

Theorem 3.4. For any $\bar{\gamma}=(\gamma_{nk})$, if $w_{nk}\leq v_{nk}$ for all but finitely many n,k, then

$$\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})\subset\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{v}).$$

Proof. Let $w_{nk}\leq v_{nk}$ for all finitely many n,k. If $\bar{x}=(x_{nk})\in\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$ then clearly $\bar{x}\in\ell^2((X,\|.\|),\bar{\gamma},\bar{v})$ because $\|\gamma_{nk}x_{nk}\|\leq 1$ for all large values of n,k. This completes the proof. □

On combining Theorems 3.1 and 3.4, we get

Theorem 3.5. For any $\bar{\gamma}=(\gamma_{nk})\,,\bar{\mu}=(\mu_{nk})\,,\bar{w}=(w_{nk})$ and $\bar{v}=(v_{nk})$ if $\liminf_{n+k\to\infty}\left|\frac{\gamma_{nk}}{\mu_{nk}}\right|^{w_{nk}}>0$, and $w_{nk}\leq v_{nk}$ for all but finitely many n,k, hold together, then

$$\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})\subset\ell^2((X,\|\cdot\|),\bar{\mu},\bar{v}).$$

Theorem 3.6. $\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$ forms a linear space over the field of complex number $\mathbb{C}$ if and only if $\bar{w}=(w_{nk})\in\ell^2_\infty$.

Proof. Let $\bar{w}=(w_{nk})\in\ell^2_\infty$ and $\bar{x}=(x_{nk})\,,\bar{y}=(y_{nk})\in\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$ then

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}x_{nk}\|^{w_{nk}}<\infty \text{ and } \lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}y_{nk}\|^{w_{nk}}<\infty.$$

Now,

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}\left(x_{nk}+y_{nk}\right)\|^{w_{nk}}\leq\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}x_{nk}\|^{w_{nk}}+\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}y_{nk}\|^{w_{nk}}<\infty.$$

Hence $\bar{x}+\bar{y}\in\ell^2((X,\|.\|),\bar{\gamma},\bar{w})$. Also, it is clear that for any scalar λ,

$$\begin{aligned}&\lambda\bar{x}\in\ell^2((X,\|.\|),\bar{\gamma},\bar{w}),\text{ since}\\ \lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\lambda\gamma_{nk}x_{nk}\|^{w_{nk}}&=\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}|\lambda|^{w_{nk}}\|\gamma_{nk}x_{nk}\|^{w_{nk}}\\ &\leq A(\lambda)\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}x_{nk}\|^{w_{nk}}<\infty.\end{aligned}$$

Conversely, if $\bar{w}=(w_{nk})\notin\ell^2_\infty$ then there exist subsequences $(n(i))$ of (n) and $(k(i))$ of (k) such that

$$w_{n(i)\ \mathrm{k(i)}}>i \text{ for each } i\geq 1. \tag{5}$$

Now taking $z_{nk}\in X$ with $\|z_{nk}\|=1$, we define a sequence $\bar{x}=(x_{nk})$ by

$$x_{nk}=\begin{cases}\gamma_{nk}^{-1}i^{-2/w_{nk}}z_{nk}, n=n(i),k=k(i),i\geq 1 \text{ and}\\ \theta,\ \text{otherwise.}\end{cases} \tag{6}$$

Then for $n=n(i),k=k(i),i\geq 1$ and in view of (5) and (6), we have

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}x_{nk}\|^{w_{nk}}=\sum_{i=1}^{\infty}\left\|\gamma_{n(i)k(i)}x_{n(i)k(i)}\right\|^{w_{n(i)k(i)}}=\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty$$

This shows that $\bar{x}$ is in $\ell^2((X,\|.\|),\bar{\gamma},\bar{w})$. But on the other hand for $n=n(i),k=k(i),i\geq 1$ and for the scalar $\lambda=4$ we have

$$\|\gamma_{nk}\lambda x_{nk}\|^{w_{nk}} = \|\gamma_{n(i)k(i)}4x_{n(i)k(i)}\|^{w_{n(i)k(i)}} = |4|^{w_{n(i)k(i)}}\frac{1}{i^2} > \frac{4^i}{i^2} > 1$$

for each $i\geq 1$, and therefore

$$\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}\lambda x_{nk}\|^{w_{nk}} > \infty,$$

which shows that

$$\lambda\bar{x}\notin \ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w}).$$

Hence $\ell^2((X,\|.\|),\bar{\gamma},\bar{w})$ is a linear space if and only if, $\bar{w}=(w_{nk})\in\ell^2_\infty$. This completes the proof. □

In the following, let $\bar{w}=(w_{nk})\in\ell^2_\infty$ and consider $\bar{x}\in\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$, we define

$$G(\bar{x}) = \lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\|\gamma_{nk}x_{nk}\|^{w_{nk}} \tag{7}$$

Theorem 3.7. Let $\bar{w}=(w_{nk})\in\ell^2_\infty$ for each $n,k\geq 1$ and X be a normed space. Then $\left(\ell^2((X,\|.\|),\bar{\gamma},\bar{w}),G\right)$ defined by (7) forms a total paranormed space.

Proof. For any $\bar{x},\bar{y}\in\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$, it can be easily verified that G satisfy following properties of paranormed space.
Clearly, $G(\bar{x})\geq 0$ and $G(\bar{x})=0$ if and only if $\bar{x}=\bar{\theta}$.

$$G(\bar{x}+\bar{y})\leq G(\bar{x})+G(\bar{y}), \text{and } G(\lambda\bar{x})\leq A(\lambda)\cdot G(\bar{x}),\ \textit{where } \lambda\in\mathbb{C}.$$

So obviously PN1, PN2 and PN3 follow.

Here we prove the continuity of scalar multiplication, i.e., PN4. For this, it suffices to prove that
(a) if $\bar{x}^{(i)}\to\bar{\theta}$ as $i\to\infty$ and $\lambda_i\to\lambda$ imply $G\left(\lambda_i\bar{x}^{(i)}\right)\to 0$ as $i\to\infty$.
(b) if $\lambda_i\to 0$ as $i\to\infty$ implies $G\left(\lambda_i\bar{x}^{(i)}\right)\to 0$ as $i\to\infty$ for each $\bar{x}\in\ell^2((X,\|.\|),\bar{\gamma},\bar{w})$.
Now to prove (a) suppose that $|\lambda_i|\leq L$ for all $i\geq 1$. Then

$$G\left(\lambda_i\bar{x}^{(i)}\right)\leq \sup_{n,k}|\lambda_i|^{w_{nk}}\lim_{K\to\infty}\sum\sum_{2\leq n+k\leq K}\left\|\gamma_{nk}x^{(i)}_{nk}\right\|^{w_{nk}}\leq A(\lambda)G\left(\bar{x}^{(i)}\right)$$

whence (a) follows.
Next if $\bar{x}\in\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w})$ then for $\varepsilon>0$ there exists I such that

$$\sum_{n+k\geq I}\|\gamma_{nk}x_{nk}\|^{w_{nk}} < \frac{\varepsilon}{2}.$$

Further if $\gamma_i\to 0$, we can find K such that when $i\geq K$, we have

$$\sum\sum_{2\leq n+k\leq i-1}|\lambda_i|^{w_{nk}}\|\gamma_{nk}x_{nk}\|^{w_{nk}} < \frac{\varepsilon}{2} \text{ and } |\alpha_i|\leq 1.$$

Thus

$$G(\lambda_i\bar{x})\leq\sum\sum_{2\leq n+k\leq i-1}\|\lambda_i\gamma_{nk}x_{nk}\|^{w_{nk}}+\sum\sum_{n+k\geq i}\|\gamma_{nk}x_{nk}\|^{w_{nk}}<\varepsilon \text{ for all } i\geq K$$

and hence (b) follows. □

Theorem 3.8. Let $\bar{w}=(w_{nk})\in\ell^2_\infty$ for each $n,k\geq 1$ and X be a Banach space. Then $\left(\ell^2((X,\|\cdot\|),\bar{\gamma},\bar{w}),G\right)$ is complete with respect to the metric $d(\bar{x},\bar{y})\leq Q(\bar{x}-\bar{y})$.

Proof. Let $\left(\bar{x}^i\right)$ be a Cauchy sequence in $\ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w})$. Thus for $0 < \varepsilon < 1$, there exists K such that

$$G\left(\bar{x}^i - \bar{x}^\ell\right) = \lim_{K \to \infty} \sum_{2 \le n+k \le K} \left\|\gamma_{nk} x_{nk}^i - \gamma_{nk} x_{nk}^\ell\right\|^{w_{nk}} < \varepsilon, \text{ for all } i, \ell \ge K.$$

Hence for each $n, k \ge 1$

$$\left\|x_{nk}^i - x_{nk}^\ell\right\| < \left|\gamma_{nk}\right|^{-1} \varepsilon^{1/w_{nk}} < \left|\gamma_{nk}\right|^{-1} \varepsilon, \text{ for all } i, \ell \ge K.$$

This shows that for each n, k, $\left(x_{nk}^i\right)_{i=1}^{\infty}$ is a Cauchy sequence in X and because of completeness of $X, x_{nk}^i \to x_{nk}$ in X, say $i \to \infty$ for each $n, k \ge 1$. Being a Cauchy sequence $\left(\bar{x}^i\right)$ is bounded, that is there exists an $L > 0$ such that for all i and $K \le 2$,

$$\sum \sum_{2 \le n+k \le K} \left\|\gamma_{nk} x_{nk}^i\right\|^{w_{nk}} \le L$$

. First taking $i \to \infty$ and then $N \to \infty$ we easily obtain that

$$\lim_{K \to \infty} \sum \sum_{2 \le n+k \le K} \left\|\gamma_{nk} x_{nk}\right\|^{w_{nk}} \le L$$

which implies that $\bar{x} = (x_{nk}) \in \ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w})$. Now for any K_1, by (3.8), we have

$$\sum \sum_{2 \le n+k \le K} \left\|\gamma_{nk} x_{nk}^i - \gamma_{nk} x_{nk}^\ell\right\|^{w_{nk}} < \varepsilon, \text{ for all } i, \ell \ge K.$$

and so letting $\ell \to \infty$ first and then $K \to \infty$, we get

$$G\left(\bar{x}^{(i)} - \bar{x}\right) = \lim_{K \to \infty} \sum \sum_{2 \le n+k \le K} \left\|\gamma_{nk} x_{nk}^{\mathrm{k}} - \gamma_{nk} x_{nk}\right\|^{w_{nk}} \le \varepsilon.$$

This shows that $\bar{x}^{\mathrm{k}} \to \bar{x}$ in $\ell^2((X, \|.\|), \bar{\gamma}, \bar{w})$ as $i \to \infty$. This proves the completeness of $\ell^2((X, \|.\|), \bar{\gamma}, \bar{w})$. □

4 Conclusion

In this paper, we have explored some conditions that characterize the linear topological structures and containment relations on double sequence space with their terms in a normed space as a generalization of the familiar sequence space. In fact, these results can be used for further generalization to investigate many other properties of the normed spaces, 2-normed spaces, and other vector-valued sequences.

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