An Insight in the Beauty of the Fibonacci

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Abstract: *Mathematics and architecture have strong logical interconnections. Ratios are good examples of their interconnectivity. Without mathematics, it is hard to believe the existence of science and arts. It is not an exaggeration to say that mathematics is everywhere. Nature is beautiful due to the proper ratios of various components in them and in relation to others. The Fibonacci sequence is one of nature's numbering systems. It is abundant in nature. It has a close relationship with the golden ratio. Golden ratios and such Fibonacci numbers are found to be used in designing logos, magazine covers, plastic surgery, to name a few. These two are two fascinating topics for mathematicians, artists, natural scientists, and philosophers. This work presents a panoramic view of the Fibonacci numbers, and the Fibonacci sequences; their mathematical presentations, patterns, properties, and beauty.*

Keywords: Fibonacci, Fibonacci sequence, Golden ratio, Beauty of Fibonacci

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1 Introduction

The Fibonacci numbers were introduced in Europe by Leonardo of Pisa, Bonacci also known as Fibonacci as his nickname through his book Liber Abaci in 1202. It is believed that these numbers were already known in Indian subcontinent much earlier. Fibonacci numbers and the Fibonacci sequence have a close connection and are found abundantly in nature. It is also related to the golden ratio and is considered as the nature's number system. It is found extensively in literature, biology, and computers [2, 9]. We begin our discussion with an introduction to the golden ratio. Consider a point X on a line segment AB with

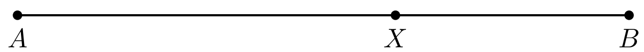

Figure 1: A line segment to get the golden ratio.

$$\frac{AB}{AX} = \frac{AX}{BX}.$$

Let $AB = c$ and $AX = 1$. Then, we get $\dfrac{c}{1} = \dfrac{1}{c-1}$, which yields $c^2 - c - 1 = 0$.

$$\therefore\ c = \frac{1 \pm \sqrt{5}}{2}.$$

For the length measure, $c = \frac{1+\sqrt{5}}{2}$, an irrational number denoted by phi (ϕ), called the golden ratio with decimal representation $\phi = 1.6180339887498948482\ldots$. It is also named as the divine ratio or divine proportion. The golden ratio and its reciprocal are two numbers whose product and difference are both equal to 1, *i.e.,*

$$\phi \,.\, \frac{1}{\phi} = 1, \text{ and } \phi - \frac{1}{\phi} = 1. \tag{1}$$

Let the first two terms of a sequence $\{f_n\}$ be $f_0 = 0$ and $f_1 = 1$. For $n \geq 2$, consider $f_n = f_{n-1} + f_{n-2}$. Then, f_n represents the n^{th} Fibonacci number with n as its index. It gives a sequence,

$$\{f_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots\}, \tag{2}$$

where the n^{th} number is the sum of two numbers preceding it. It is also found to be started with 1 instead of 0. These numbers and the respective sequence in Equation (2) are the Fibonacci numbers and the Fibonacci sequence, respectively. Here the ratio of a term to its previous term, say, $\frac{610}{377} = 1.61803\ldots$, and this ratio gets closer to the golden ratio ϕ. This has been shown in Table 1. For details, we refer to [2, 4, 6, 9].

Table 1: Fibonacci sequence and the ratio for $\dfrac{f_{n+1}}{f_n}$.

S.No.	Fibonacci sequence	Ratio for $\frac{f_{n+1}}{f_n}$
1	1	1.000000
2	2	2.000000
3	3	1.500000
4	5	1.666667
5	8	1.600000
6	13	1.625000
7	21	1.615384
8	34	1.619048
9	55	1.617647
10	89	1.618182
11	144	1.617977
12	233	1.618055
13	377	1.618025
14	610	1.618037
15	987	1.618033
16	1597	1.618034
17	2584	1.618034

The Fibonacci sequence is one of the fascinating topics from ancient history. There has been a lot of work about its historical background and existence. However, its systematic overview from the mathematical perspective is somewhat lacking. This paper presents the state-of-art on the Fibonacci sequence, their mathematical structures, construction, properties, beauty, and applications in diversified fields. Section 2 is about Fibonacci sequence. Section 3 presents its properties. Section 4 discusses its existence and applications. Finally, Section 5 concludes the paper.

2 Fibonacci Sequence

In the 12^{th} century, the Italian Mathematician Leonardo Pisano used a series to illustrate a problem based on a pair of breeding rabbits, as a Fiabonacci series, as mentioned as a Fibonacci puzzle as in Section 2.1 as illustrated in Knott [7]. For details, we refer to the classical book like Dunlap [2].

2.1 Fibonacci puzzle

Fibonacci puzzle deals with a biologically unrealistic scenario, where the population growth of an idealized rabbit family is considered. It is assumed that a newly born pair of rabbits (one male and the other female) are kept in a field, and each breeding pair reach sexual maturity and always mate at their age of one month. Let at the end of their second month, they always produce another pair of rabbits. It is also assumed that the rabbits continue living and breeding forever. Then, the following discussion shows how many pairs there will be in one year.

- As at the end of the 1^{st} month, they just mate, there are still 1 pair.
- Now, at the end of the 2^{nd} month, they produce a new pair. So, there are 2 pairs of rabbits in the field.
- At the end of the 3^{rd} month, the original pair give birth to a 2^{nd} pair. So, there are 3 pairs in total.

- And, at the end of the 4^{th} month, the original pair along with the new pair born two months ago can produce their new pairs of offspring. Hence, there will be 5 pairs, and so on.

Thus, the number of the pair of rabbits at the end of the twelve months can respectively be expressed as: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 98, and 153, thereby forming a Fibonacci sequence.

2.2 Fibonacci tiling

Fibonacci tiling is one of the beautiful tiling patterns. It can be constructed by using the Fibonacci sequence. Its construction pattern is similar to the construction of the golden spiral [2] as in Fig. 2.

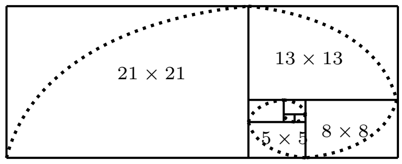

Figure 2: Fibonacci tiling.

3 Properties of Fibonacci Sequence

Fibonacci sequence has several interesting properties. Here, we present its relationship with Pascal's coefficients, interconnection with ϕ, recurring formula, the general term, and the Fibonacci prime.

3.1 Relationship with Pascal's coefficients

Sum of the elements in each diagonal of the Pascal's triangle is equal to the corresponding Fibonacci sequence term, *i.e.*, $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = f_{n+1}$. Here, as in Pascal's coefficients given in Fig. 3, the numbers in red are 1, 1, 2, 3, 5, 8, 13, 34, 55, 89, $\cdots$ forming a Fibonacci sequence.

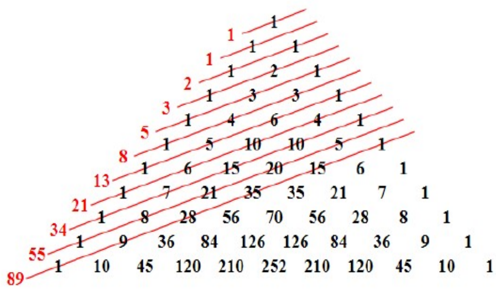

Figure 3: Pascal's coefficients.

3.2 Interconnection with golden ratio

The Fibonacci sequence $\{f_n\}$ and the golden ratio ϕ are interconnected. This can be illustrated with the help of following example.

Example 1. *If we raise the ϕ to consecutive powers, a familiar pattern emerge to Fibonacci [2].*

In fact,

$$\begin{aligned}
\phi &= 1.618034\ldots \\
\phi^2 &= 2.618034\cdots = 1+\phi \\
\phi^3 &= \phi^2.\phi = (1+\phi).\phi = \phi+\phi^2 = \phi+1+\phi = 1+2\phi \\
\phi^4 &= \phi^3.\phi = (1+2\phi).\phi = \phi+2\phi^2 = \phi+2(1+\phi) = \phi+2+2\phi = 2+3\phi \\
\phi^5 &= \phi^4.\phi = (2+3\phi).\phi = 2\phi+3\phi^2 = 2\phi+3(1+\phi) = 2\phi+3+3\phi = 3+5\phi \\
\phi^6 &= \phi^5.\phi = (3+5\phi).\phi = 3\phi+5\phi^2 = 3\phi+5(1+\phi) = 3\phi+5+5\phi = 5+8\phi
\end{aligned}$$

It emerges to Fibonacci, 1, 1, 2, 3, 5, 8,... and can be represented as in Equation 3.

$$\phi^n = f_{n-1} + f_n.\phi. \tag{3}$$

Hence, we have the following lemmas:

Lemma 1. *[2]. If $\{f_n\}$ is a Fibonacci sequence and ϕ is the golden ratio, then they are related by the following different relations.*

a) $\phi^n = \phi.f_n + f_{n-1}$.

b) $f_n = \frac{1}{2\phi-1}[\phi^n - (1-\phi)^n]$.

c) $\phi = \frac{1+\phi}{\phi} + \frac{1+2\phi}{1+\phi} + \frac{1+3\phi}{1+2\phi} + \frac{3+5\phi}{2+3\phi} + \cdots + \frac{f_n+f_{n+1}\phi}{f_{n-1}+f_n\phi} \approx 1.618034.$

d) $\frac{f_n+\phi.f_{n+1}}{f_{n-1}+\phi.f_n} = \cdots = \frac{2\phi-3}{5-3\phi} + \frac{2-\phi}{2\phi-3} + \frac{\phi-1}{2\phi} + \frac{1}{\phi-1} = \frac{\phi}{1} \approx 1.618034.$

3.3 Recurring formula

Let $f_0 = 1$ and $f_1 = 1$. Then the Fibonacci sequence $\{f_n\}$ can be expressed as the second order difference equation as, $f_{n+2} - f_{n+1} - f_n = 0$, equivalent to the recurring formula,

$$f_{n+2} = f_{n+1} + f_n. \tag{4}$$

Finding the seventh term of a Fibonacci sequence is not so difficult task but to find seventieth term is much more cumbersome, as it demands the sum the consecutive pairs of the previous sixty-nine terms.

Alternating form of the recurring formula: From Equation (4), we can write, $f_{n+2} = f_{n+1} + f_n$, equivalently, $f_n = f_{n-1} + f_{n-2}$.
Assume $f_n = r^n$ be the solution.
Then, $r^n = r^{n-1} + r^{n-2} \implies r^2 = r + 1 \implies r^2 - r - 1 = 0$, *i.e.*, $r = \frac{1\pm\sqrt{5}}{2}$.
Let $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
Hence, $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n \implies f_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$.
But, $f_0 = \alpha_1 + \alpha_2 = 0$ and $f_1 = \alpha_1(\frac{1+\sqrt{5}}{2}) + \alpha_2(\frac{1-\sqrt{5}}{2}) = 1 \implies \alpha_1 = -\alpha_2$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$.

$$\therefore f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(\frac{-1}{\phi}\right)^n\right]. \tag{5}$$

For details, we refer to Rosen [11].

Theorem 1. *[10] Let f_n and f_{n+1} are two successive Fibonacci numbers, then*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi. \tag{6}$$

Proof.

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}[\phi^{n+1} - (\frac{-1}{\phi})^{n+1}]}{\frac{1}{\sqrt{5}}[\phi^n - (\frac{-1}{\phi})^n]} = \lim_{n\to\infty} \frac{\phi[\phi^n - \frac{1}{\phi}(\frac{-1}{\phi})^{n+1}]}{\phi^n - (\frac{-1}{\phi})^n}$$

$$= \phi. \lim_{n\to\infty} \frac{\phi^n + (\frac{-1}{\phi})^{n+2}}{\phi^n - (\frac{-1}{\phi})^n} . \frac{\frac{1}{\phi^n}}{\frac{1}{\phi^n}} = \phi. \lim_{n\to\infty} \frac{1 + \frac{(-1)^{n+2}}{\phi^{n+2}}}{1 - \frac{(-1)^n}{\phi^{2n}}} = \phi. \frac{1+0}{1-0} = \phi.$$

□

3.4 Beauty of the Fibonacci

Let the Fibonacci sequence be $\{f_n\}$= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, $\cdots$}. Below, we present some interesting and beautiful features of the Fibonacci numbers and the sequence [3, 10].

1. Double any Fibonacci number and subtract the latter number to get the number two places before the original.
   e.g., $2f_4 - f_5 = 2 \times 3 - 5 = 1 = f_2$,
   $2f_5 - f_6 = 2 \times 5 - 8 = 2 = f_3$,
   $2f_6 - f_7 = 2 \times 8 - 13 = 3 = f_5$, and so on.
   In general, $2f_n - f_{n+1} = f_{n-2}$.

2. Starting with the first odd-positioned number, add the consecutive odd-positioned Fibonacci, it gives the latter term to the last term in the sum.
   e.g., $f_1 + f_3 = 1 + 2 = 3 = f_4$,
   $f_1 + f_3 + f_5 = 1 + 2 + 5 = 8 = f_6$,
   $f_1 + f_3 + f_5 + f_7 = 1 + 2 + 5 + 13 = 21 = f_8$, and so on.
   In general, $f_1 + f_3 + f_5 + f_7 + \cdots + f_{2n-1} = f_{2n}$.

3. The product of any term of the Fibonacci sequence and the term after two places results in one less or more than the square of the term between the two. More precisely, if the squared number is at the even position of the Fibonacci sequence, then one is added. On contrary, if it is odd-positioned, then one is subtracted.
   e.g., $f_4 \times f_6 = 3 \times 8 = 24$ and $f_5^2 = 5^2 = 25$,
   $f_7 \times f_9 = 13.34 = 442$ and $f_8^2 = 21^2 = 441$, *etc.*
   In general, $f_{n-1} \times f_{n+1} = f_n^2 \pm 1$.

4. If the square of a term in a Fibonacci sequence is subtracted from the square of the term two places after it, then it again results in a Fibonacci number, indexed by the sum of those of the numbers.
   e.g., $f_5^2 - f_3^2 = 5^2 - 2^2 = 21 = f_8$,
   $f_6^2 - f_4^2 = 8^2 - 3^2 = 55 = f_{10}$,
   $f_7^2 - f_5^2 = 13^2 - 5^2 = 144 = f_{12}$, *etc.*
   In general, $f_n^2 - f_{n-2}^2 = f_{2n-2}$.

5. On adding the squares of the two consecutive Fibonacci numbers again yield a Fibonacci number, indexed by the sum of those of the numbers, with the index by the sum of those of the numbers.
   e.g., $f_3^2 + f_4^2 = 2^2 + 3^2 = 13 = f_7$
   $f_4^2 + f_5^2 = 3^2 + 5^2 = 34 = f_9$
   $f_5^2 + f_6^2 = 5^2 + 8^2 = 89 = f_{11}$, *etc.*
   In general, $f_n^2 + f_{n+1}^2 = f_{2n+1}$.

6. When any two consecutive terms of a Fibonacci sequence are squared and added, it again results in a Fibonacci number. In fact, they again form a sequence of alternate Fibonacci numbers starting from f_3. In fact, $1^2 + 1^2 = 2, 1^2 + 2^2 = 5, 2^2 + 3^2 = 13$, with the alternate Fibonacci, $2, 5, 13, \ldots$.

7. Among any four consecutive Fibonacci numbers, the difference of the squares of the two means is equal to the product of the two extremes.
   In particular, for f_2, f_3, f_4, f_5, we have $f_4^2 - f_3^2 = 5^2 - 3^2 = 16 = f_2 \times f_5$.
   In general, $f_{n+1}^2 - f_n^2 = f_{n-1} \times f_{n+2}$.

8. Starting with f_1, the addition of the squares of the consecutive terms up to a term in the Fibonacci sequence produces a number which can be expressed as a product of two consecutive Fibonacci numbers: the last number that is squared and the Fibonacci number that immediately follows.
   e.g., $1^2 + 1^2 = 2 = 1 \times 2$
   $1^2 + 1^2 + 2^2 = 6 = 2 \times 3$
   $1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5$
   $1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \times 8$, *etc.*
   In general, $\sum_{i=1}^{k} f_i^2 = f_k \times f_{k+1}$.

9. Among any three consecutive Fibonacci numbers, subtraction of the cube of the smallest one from the sum of the cubes of the two greater again yields a Fibonacci number, indexed by the sum of those of the numbers involved.
   e.g., $f_3^3 + f_4^3 - f_2^3 = 2^3 + 3^3 - 1^3 = 8 + 23 - 1 = 34 = f_9$
   $f_5^3 + f_6^3 - f_4^3 = 5^3 + 8^3 - 3^3 = 125 + 512 - 27 = 610 = f_{15}$, *etc.*
   In general, $f_{n+1}^3 + f_{n+2}^3 - f_n^3 = f_{3n+3}$.

10. When any finite number of consecutive Fibonacci numbers are added, it results in a number that is one less than a Fibonacci number as the next two places of the last one added, *i.e.,* $f_1 + f_2 + f_3 + f_4 + f_5 = 1 + 1 + 2 + 3 + 5 = 12 = 13 - 1 = f_7 - 1$. Hence, $\sum_{i=1}^{k} f_i = f_{k+2} - 1$.

11. The sum of any ten consecutive Fibonacci numbers is divisible by 11, *i.e.,* $11|(f_n + f_{n+1} + \cdots + f_{n+9})$.

12. If q divides p, then f_q divides f_q, *i.e.*, $q|p \implies f_q|f_p$, for any $p, q \in Z^+$.

13. It is a general observation that any two consecutive terms of a Fibonacci sequence are relatively prime, *i.e.*, g.c.d. $(f_n, f_{n+1}) = 1$.

3.5 A Fibonacci prime

A prime number in a Fibonacci sequence is a Fibonacci prime. The first few such numbers are $2, 3, 5, 13, 89, 233, 1597, 28657$. It is still an open question, whether there are infinitely many Fibonacci primes [10].

4 Fibonacci: The Nature's Numbering System

Fibonacci numbers and the corresponding sequence is one of the nature's numbering systems. It is found everywhere in nature and in various other fields.

4.1 Nature

Fibonacci numbers also appear beautifully in nature in the spiral growth patterns like the leaf arrangements in plants, the number of spirals on a cactus, or in sunflowers seedbeds. It has a close connection to the golden ratio too. Steep and the gradual spirals up the side of pine cones are almost counted as Fibonacci numbers. Examples include: some pine cones have three gradual and five steep spirals, whereas some others have eight gradual and thirteen steep spirals. Moreover, in a bee hive, there are generally three types of bees. The queen bee lays eggs, the male bees have no specific works, whereas the female bees do all works there [10]. Male bees are developed from unfertilized eggs so they have only mothers no father but the female bees are developed from the eggs which are fertilized. So they do have both the parents. So one male bee has 1 mother, 2 grandparents, 3 great-grandparents, 3 great-great-grandparents, and 8 great-great-great-grandparents. Interestingly, in each preceding generation, the number of bees is a Fibonacci number [5].

The petals of many flowers are found to be arranged to form a Fibonacci number. For example, some

Figure 4: Different flowers having arrangements in petals to Fibonacci number.
http://www.google.com/goldennumber.net

flowers along with the number of petals are: White Calla Lily-1, Euphorbia-2, Trillium-3, Columbine-5, Bloodroot-8, Black-Eyed Susan-13, Shasta Daisy-21, and Field Daisies-34, as in Figure 4, as mentioned in Akhtaruzzaman and Shafie [1].

4.2 Music

Let a song last for 4 minutes. Then it is divided into two parts, at 61.8% and 38.2%, with a modification by a certain change: a bridge, or an arrangement with a different instrument or with a new melodic composition at 61.8% of 240 seconds, *i.e.,* at 148.32 seconds. The changes in the rhythm of the song are followed on the Fibonacci sequence, to make it lovelier to listen to. Great composers of western music, to name a few, Mozart, Beethoven, and Wagner deliberately changed the rhythm of their music in sequences. In music, octave refers to the eight whole tones of the complete musical scale. And, there are 13 notes in the span of any notes through its octave. The dominant note is the 5^{th} note of the major scale that is the 8^{th} note of all the 13 notes on the octave. The key frequencies of the musical notes are related to $1, 1, 2, 3, 5, \text{and}, 8$ as some Fibonacci numbers. Musical compositions often reflect these numbers and the golden ratio on their timing. Musical notes progress in high and low pitches with an infinite spiral in the same manner as the golden spiral. The golden ratio is found to be the mathematical translation of an algorithm used by nature and is a lesson on aesthetic perfection, beauty, harmony, and pleasure to the music too. Various musical instruments, such as guitars, piano, violin, and even in the high-quality speaker wire, are found to have used the golden ratio in their designs. For details, we refer to Akhtaruzzaman and Shafie[1] and Tamargo et *al.* [12].

4.3 Poetry

Haiku is a three-line poetry grounded in mathematical constraints with five, seven, and five syllables in three consecutive lines. Based in this, the Fib, known as the Fibonacci poem is written. The Fib poetry is in a structured form with 1, 1, 2, 3, 5, and 8 syllables, respectively, [8]. Below, we compose a couple of pieces of Fib poetry.

Sea
says
to fish,
" Leave alone;
please forget my heart."

Just
same
You breathe
Saying ever
Oh, Sambodhika !
Needless to say, I remember

5 Conclusion

Fibonacci numbers and the corresponding Fibonacci sequence form nature's numbering systems. The Fibonacci sequence has a close relationship with the golden ratio ϕ. In present days, golden ratios and such Fibonacci are best used in designing logos, magazine covers, plastic surgery, etc. This work has presented a panoramic view of the Fibonacci numbers, and the Fibonacci sequences; their patterns, and properties and opens a wide horizon for the further research.

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