Positive Operator Frame for Hilbert C^*-modules

Hatim Labrigui[1], Hafida Massit[1], Mohamed Rossafi[2,*]

[1]Laboratory Partial Differential Equations, Spectral Algebra and Geometry, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, P. O. Box 133 Kenitra, Morocco
[3]LaSMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, P. O. Box 1796 Fez Atlas, Morocco

*Correspondence to: Mohamed Rossafi, Email: rossafimohamed@gmail.com; mohamed.rossafi@usmba.ac.ma

Abstract: *The work on frame theory has undergone a remarkable evolution over the last century. Several related properties have applications on many fields of mathematics, engineering, signal and image processing, informatics, medecine and probability. In order to search for new results related to the role of operators in frame theory using the characterization of the positive elements in a C^*-algebra, we introduce the concept of positive operator frame, L-positive operator frame, $*$-positive operator frame and $*$-L-positive operator frame for the set of all adjointable operators on a Hilbert C^*-module denoted $End^*_{\mathcal{B}}(H)$ where L is a positive operator. Also, we give some new properties.*

Keywords: Frame, Positive operator, C^*-algebra, Hilbert C^*-modules

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1 Introduction and Preliminaries

In 1952, frame theory was introduced by Duffin and Schaefer [7] to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [9] for signal processing.

In the year 2000, M. Frank and D. R. Larson [8] have extended the theory for the elements of C^*-algebra and Hilbert C^*-modules. Eventually, frames with C^*-valued bounds in Hilbert C^*-modules have been considered in [1]. The basic idea was to consider module over C^*-algebra instead of linear spaces and to allow the inner product to take values in the C^*-algebra. After the fundamental paper [5] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [9].

During this century, frame theory developed rapidly and experienced several generalizations either in Hilbert spaces or in Hilbert C^*-modules. For more details see [10, 13, 14, 15, 16, 17, 18, 19].

In this paper, we introduce the concept of positive operator frame, L-positive operator frame, $*$-positive operator frame and $*$-L-positive operator frame for the set of all adjointable operators on a Hilbert C^*-module denoted $End^*_{\mathcal{B}}(\mathcal{V})$ where L is a positive operator. Also, we establish some new properties. Some illustrative examples are provided to advocate the usability of our results.

In the following we briefly recall the definitions and basic properties of Hilbert $\mathcal{B}$-module, where $\mathcal{B}$ is a C^*-algebra. Our references for C^*-algebras are [4, 6].

Definition 1.1. [4]. Let $\mathcal{B}$ be a unital C^*-algebra and $\mathcal{V}$ be a left $\mathcal{B}$-module, such that the linear structures of $\mathcal{B}$ and $\mathcal{V}$ are compatible. $\mathcal{V}$ is a pre-Hilbert $\mathcal{B}$-module if $\mathcal{V}$ is equipped with an $\mathcal{B}$-valued inner product $\langle .,.\rangle_{\mathcal{B}} : \mathcal{V} \times \mathcal{V} \to \mathcal{B}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle f, f\rangle_{\mathcal{B}} \geq 0$, for all $f \in \mathcal{V}$, and $\langle f, f\rangle_{\mathcal{B}} = 0$ if and only if $f = 0$.

(ii) $\langle \alpha f + g, h\rangle_{\mathcal{B}} = \alpha\langle f, h\rangle_{\mathcal{B}} + \langle g, h\rangle_{\mathcal{B}}$, for all $\alpha \in \mathcal{B}$ and $f, g, h \in \mathcal{V}$.

(iii) $\langle f, g\rangle_{\mathcal{B}} = \langle g, f\rangle^*_{\mathcal{B}}$, for all $f, g \in \mathcal{V}$.

For $f \in \mathcal{V}$, we define $||f|| = ||\langle f, f\rangle_{\mathcal{B}}||^{\frac{1}{2}}$. If $\mathcal{V}$ is complete with $||.||$, it is called a Hilbert $\mathcal{B}$-module or a Hilbert C^*-module over $\mathcal{B}$.
For every b in C^*-algebra $\mathcal{B}$, we have $|b| = (b^*b)^{\frac{1}{2}}$ and the $\mathcal{B}$-valued norm on $\mathcal{V}$ is defined by $|f| = \langle f, f\rangle_{\mathcal{B}}^{\frac{1}{2}}$, for all $f \in \mathcal{V}$.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert $\mathcal{B}$-modules, a map $L : \mathcal{V}_1 \to \mathcal{V}_2$ is said to be adjointable if there exists a map $L^* : \mathcal{V}_2 \to \mathcal{V}_1$ such that $\langle Lf, g\rangle_{\mathcal{B}} = \langle f, L^*g\rangle_{\mathcal{B}}$ for all $f \in \mathcal{V}_1$ and $g \in \mathcal{V}_2$.

We reserve the notation $End^*_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_2)$ for the set of all adjointable operators from $\mathcal{V}_1$ to $\mathcal{V}_2$ and $End^*_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_1)$ is abbreviated to $End^*_{\mathcal{B}}(\mathcal{V}_1)$.

In what follows, let $\mathcal{I}$ and $\mathcal{J}$ be finite or countable index subset of $\mathbb{N}$.

We will give lemmas and definitions that will be useful to prove the results of this work.

Lemma 1.2. *[1]. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert $\mathcal{B}$-modules and $L \in End^*_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_2)$.*

(i) If L is injective and L has the closed range, then the adjointable map L^*L is invertible and

$$\|(L^*L)^{-1}\|^{-1} I_{\mathcal{V}_1} \leq L^*L \leq \|L\|^2 I_{\mathcal{V}_1}.$$

(ii) If L is surjective, then the adjointable map LL^* is invertible and

$$\|(LL^*)^{-1}\|^{-1} I_{\mathcal{V}_2} \leq LL^* \leq \|L\|^2 I_{\mathcal{V}_2}.$$

Lemma 1.3. *[12]. Let $\mathcal{V}$ be an Hilbert $\mathcal{B}$-module. If $L \in End^*_{\mathcal{B}}(\mathcal{V})$, then*

$$\langle Lf, Lf\rangle_{\mathcal{B}} \leq \|L\|^2 \langle f, f\rangle_{\mathcal{B}}, \qquad f \in \mathcal{V}.$$

Lemma 1.4. *[2]. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert $\mathcal{B}$-modules and $L \in End^*_{\mathcal{B}}(\mathcal{V}_1, \mathcal{V}_2)$. Then the following statements are equivalent,*

(i) L is surjective.

(ii) L^* is bounded below with respect to norm, i.e., there is $m_1 > 0$ such that $\|L^*f\| \geq m_1\|f\|$, for all $f \in \mathcal{V}_2$.

(iii) L^* is bounded below with respect to the inner product, i.e., there is $m_2 > 0$ such that $\langle L^*f, L^*f\rangle_{\mathcal{B}} \geq m_2\langle f, f\rangle_{\mathcal{B}}$, for all $f \in \mathcal{V}_2$.

Definition 1.5. Let $L \in End^*_{\mathcal{B}}(\mathcal{V})$, where $\mathcal{V}$ is a Hilbert $\mathcal{B}$-module. Then L is called a positive operator if we have $\langle Lf, f\rangle_{\mathcal{B}} \geq 0$ for all $f \in \mathcal{V}$.
The set of all positive adjointable operators is denoted $End^{*^+}_{\mathcal{B}}(\mathcal{V})$.

Definition 1.6. [21] A sequence $\{\Phi_i \in End^*_{\mathcal{B}}(\mathcal{V}, \mathcal{V}_i) : i \in \mathcal{I}\}$ is called a g-frame for a Hilbert C^*-module $\mathcal{V}$ with respect to $\{\mathcal{V}_i : i \in \mathcal{I}\}$ if there exist constants $E, F > 0$ such that

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, \Phi_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{1.1}$$

The constants E and F are called the bounds of $\{\Phi_i\}_{i\in\mathcal{I}}$. If $E = F = \alpha \neq 1$, the g-frame is called α-tight. If only the right side of the inequality (1.1) holds, we call it a g-Bessel sequence.

Definition 1.7. [3] Let $L \in End^*_{\mathcal{B}}(\mathcal{V})$. A sequence $\{\Phi_i \in End^*_{\mathcal{B}}(\mathcal{V}, \mathcal{V}_i) : i \in \mathcal{I}\}$ is called a L-g-frame for a Hilbert C^*-module $\mathcal{V}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, if there exist constants $E, F > 0$ such that

$$E\langle L^*f, L^*f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, \Phi_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{1.2}$$

The constants E and F are called the bounds for $\{\Phi_i\}_{i\in\mathcal{I}}$.
A L-g-frame $\{\Phi_i\}_{i\in\mathcal{I}}$ is said to be tight if there exists a constant $E > 0$ such that

$$E\langle L^*f, L^*f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle \Phi_i f, \Phi_i f\rangle_{\mathcal{B}}, \quad f \in \mathcal{V}. \tag{1.3}$$

Definition 1.8. [11] A sequence $\{L_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ is said to be an operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, if there exist constants $E, F > 0$ such that

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, L_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{1.4}$$

where E and F are called the bounds for the operator frame. An operator frame $\{L_i\}_{i\in\mathcal{I}}$ is said to be tight if $E = F \neq 1$. If only upper inequality of (1.4) holds, then $\{L_i\}_{i\in\mathcal{I}}$ is called an operator Bessel sequence for $End^*_{\mathcal{B}}(\mathcal{V})$.

Definition 1.9. [20] Let $L \in End^*_{\mathcal{B}}(\mathcal{V})$. A sequence $\{L_i\}_{i\in\mathcal{I}}$ on a Hilbert C^*-module $\mathcal{V}$ is said to be a L-operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, if there exist constants $E, F > 0$ such that

$$E\langle L^* f, L^* f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, L_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}, \tag{1.5}$$

where E and F are called the bounds of $\{L_i\}_{i\in\mathcal{I}}$. A L-operator frame $\{L_i\}_{i\in\mathcal{I}}$ is said to be tight if there exists a constant $E > 0$ such that

$$E\langle L^* f, L^* f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle L_i f, L_i f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{1.6}$$

2 Positive Operator Frame

Definition 2.1. A sequence of adjointable operators $\{L_i\}_{i\in\mathcal{I}}$ on a Hilbert C^*-module $\mathcal{V}$ is said to be a positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, if there exist constants $E, F > 0$ such that

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{2.1}$$

The numbers E and F are called the bounds of $\{L_i\}_{i\in\mathcal{I}}$. If $E = F$, the positive operator frame is called tight. If $E = F = 1$, it is called a normalized tight positive operator frame or a Parseval positive operator frame. If only upper inequality of (5.1) holds, then $\{L_i\}_{i\in\mathcal{I}}$ is called a Bessel positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$.

Example 2.2. Let $\mathcal{V}$ be a Hilbert C^*-module and $(f_i)_{i\in\mathcal{I}}$ be a frame for $\mathcal{V}$, then there exist constants $E, F > 0$ such that

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle f, f_i\rangle_{\mathcal{B}} \langle f_i, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}. \tag{2.2}$$

We define the sequence of operators $\{L_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ by $L_i = f_i \otimes f_i,\ i \in \mathcal{I}$, so we have for all $f \in \mathcal{V}$,

$$\sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle (f_i \otimes f_i) f, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle \langle f, f_i\rangle_{\mathcal{B}} f_i, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle f, f_i\rangle_{\mathcal{B}} \langle f_i, f\rangle_{\mathcal{B}},$$

Then by (2.2), we have

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}.$$

Thus $\{L_i\}_{i\in\mathcal{I}}$ is a positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$.

Theorem 2.3. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. Then $\{L_i\}_{i\in\mathcal{I}}$ corresponds to an operator frame. The converse is also valid.*

Proof. Let $\{R_i\}_{i\in\mathcal{I}}$ be an operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, then there exist constants $E, F > 0$ such that

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle R_i f, R_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V},$$

so

$$E\langle f,f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle R_i^*R_if,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}; \quad f\in\mathcal{V}.$$

Thus $\{R_i^*R_i\}_{i\in\mathcal{I}}$ is a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$.

For the converse, let $\{\tilde{R}_i\}_{i\in\mathcal{I}}$ be a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$.

Since $\tilde{R}_i$ is a positive element in $End_{\mathcal{B}}^*(\mathcal{V})$, then there exist $R_i \in End_{\mathcal{B}}^*(\mathcal{V})$ such that $\tilde{R}_i = R_i^*R_i \quad i\in\mathcal{I}$.

$\{\tilde{R}_i\}_{i\in\mathcal{I}}$ is a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. Then there exist constants $E,F>0$ such that

$$E\langle f,f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle \tilde{R}_if,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}} \quad f\in\mathcal{V},$$

so

$$E\langle f,f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle R_i^*R_if,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}} \quad f\in\mathcal{V},$$

thus

$$E\langle f,f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle R_if,R_if\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}} \quad f\in\mathcal{V},$$

hence $\{R_i\}_{i\in\mathcal{I}}$ is an operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. □

Corollary 2.4. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a positive tight operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. The sequence $\{L_i\}_{i\in\mathcal{I}}$ corresponds to a tight operator frame. The converse is also valid.*

Corollary 2.5. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a Bessel positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. The sequence $\{L_i\}_{i\in\mathcal{I}}$ corresponds to an operator Bessel sequence. The converse is valid.*

Theorem 2.6. *Let $\{\Phi_i \in End_{\mathcal{B}}^*(\mathcal{V},\mathcal{V}_i) : i\in\mathcal{I}\}$ be a g-frame for $\mathcal{V}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, then the sequence $\{\Phi_i\}_{i\in\mathcal{I}}$ corresponds to a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. The converse is valid.*

Proof. Results from the properties of the positive elements in a C^*-algebra. □

Definition 2.7. Let $\{L_i\}_{i\in\mathcal{I}}$ be a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. The operator S_L defined by

$$\begin{aligned} S_L : \mathcal{V} &\longrightarrow \mathcal{V} \\ f &\longmapsto S_Lf = \sum_{i\in\mathcal{I}} L_if. \end{aligned}$$

is called the frame operator associated to $\{L_i\}_{i\in\mathcal{I}}$ or frame operator only if there is no confusion.

Proposition 2.8. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$ with bounds E and F and frame operator S_L, then the operator S_L is positive, self-adjoint and invertible. Moreover, we have $EI_{\mathcal{V}} \leq S_L \leq FI_{\mathcal{V}}$ and the following equality,*

$$f = \sum_{i\in\mathcal{I}} S_L^{-1}L_if = \sum_{i\in\mathcal{I}} L_iS_L^{-1}f; \quad f\in\mathcal{V}, \tag{2.3}$$

called the reconstruction formula.

Proof. From the definition of frame operator, it's easy to show that the operator S_L is positive and self-adjoint.

Moreover, we have

$$\langle Ef,f\rangle_{\mathcal{B}} = E\langle f,f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle L_if,f\rangle_{\mathcal{B}} = \langle S_Lf,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}} = \langle f,f\rangle_{\mathcal{B}}.$$

This shows that

$$EI_{\mathcal{V}} \leq S_L \leq FI_{\mathcal{V}},$$

which implies that S_L is invertible. Further, for any $f \in \mathcal{V}$, we have

$$f = S_L^{-1} S_L f = S_L^{-1} \sum_{i\in\mathcal{I}} L_i f = \sum_{i\in\mathcal{I}} S_L^{-1} L_i f,$$

and

$$f = S_L S_L^{-1} f = \sum_{i\in\mathcal{I}} L_i S_L^{-1} f.$$

□

Theorem 2.9. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert C^*-modules over a C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ respectively. Let $\{\Phi_i\}_{i\in\mathcal{I}}$ be a positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V}_1)$ with bounds E, F and frame operator S_Φ. Let $\{\Psi_j\}_{j\in\mathcal{J}}$ be a positive operator frame for $End_{\mathcal{A}}^*(\mathcal{V}_2)$ with bounds M, N and frame operator S_Ψ. Then $\{\Phi_i \otimes \Psi_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a positive operator frame for $End_{\mathcal{B}\otimes\mathcal{A}}^*(\mathcal{V}_1 \otimes \mathcal{V}_2)$ with bounds EM and FN and frame operator $S_{\Phi\otimes\Psi} = S_\Phi \otimes S_\Psi$.*

Proof. From the definition of $\{\Phi_i\}_{i\in\mathcal{I}}$ and $\{\Psi_j\}_{j\in\mathcal{J}}$ we have,

$$E\langle f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}, \quad f \in \mathcal{V}_1.$$

$$M\langle g, g\rangle_{\mathcal{A}} \leq \sum_{j\in\mathcal{J}} \langle \Psi_j g, g\rangle_{\mathcal{A}} \leq N\langle g, g\rangle_{\mathcal{A}} \quad g \in \mathcal{V}_2.$$

Therefore,

$$\begin{aligned} EM\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}} &\leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \sum_{j\in\mathcal{J}} \langle \Psi_j g, g\rangle_{\mathcal{A}} \\ &\leq FN\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}}, \quad f \in \mathcal{V}_1, \quad g \in \mathcal{V}_2. \end{aligned}$$

Then,

$$\begin{aligned} EM\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} &\leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \langle \Psi_j g, g\rangle_{\mathcal{A}} \\ &\leq FN\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}, \end{aligned}$$

Consequently, for all $f \otimes g \in \mathcal{V}_1 \otimes \mathcal{V}_2$ we have:

$$\begin{aligned} EM\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} &\leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f \otimes \Psi_j g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \\ &\leq FN\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}. \end{aligned}$$

Then,

$$\begin{aligned} EM\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} &\leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle (\Phi_i \otimes \Psi_j)(f \otimes g), f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \\ &\leq FN\langle f \otimes g, f \otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}. \end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{V}_1 \otimes_{alg} \mathcal{V}_2$ and then it's satisfied for all $h \in \mathcal{V}_1 \otimes \mathcal{V}_2$. It shows that $\{\Phi_i \otimes \Psi_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a positive operator frame for $End_{\mathcal{B}\otimes\mathcal{A}}^*(\mathcal{V}_1 \otimes \mathcal{V}_2)$ with bounds EM and FN, respectively.

By the definition of frame operator S_Φ and S_Ψ, we have

$$S_\Phi f = \sum_{i\in\mathcal{I}} \Phi_i f, \quad f \in \mathcal{V}_1.$$

$$S_\Psi g = \sum_{j\in\mathcal{J}} \Psi_j g, \quad g \in \mathcal{V}_2.$$

Therefore:

$$\begin{aligned}(S_\Phi \otimes S_\Psi)(f \otimes g) &= S_\Phi f \otimes S_\Psi g \\ &= \sum_{i\in\mathcal{I}} \Phi_i f \otimes \sum_{j\in\mathcal{J}} \Psi_j g \\ &= \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \Phi_i f \otimes \Psi_j g \\ &= \sum_{i\in\mathcal{I}, j\in\mathcal{J}} (\Phi_i \otimes \Psi_j)(f \otimes g).\end{aligned}$$

By the uniqueness of frame operator, the last expression is equal to $S_{\Phi\otimes\Psi}(f \otimes g)$, consequently we have $(S_\Phi \otimes S_\Psi)(f \otimes g) = S_{\Phi\otimes\Psi}(f \otimes g)$. The last equality is satisfied for every finite sum of elements in $\mathcal{V}_1 \otimes_{alg} \mathcal{V}_2$ and then it's satisfied for all $h \in \mathcal{V}_1 \otimes \mathcal{V}_2$. It shows that $(S_\Phi \otimes S_\Psi)(h) = S_{\Phi\otimes\Psi}(h)$, so $S_{\Phi\otimes\Psi} = S_\Phi \otimes S_\Psi$. □

3 Positive K-operator Frame

Definition 3.1. Let $L \in End_{\mathcal{B}}^{*^+}(\mathcal{V})$. A family of adjointable operators $\{L_i\}_{i\in\mathcal{I}}$ on a Hilbert C^*-module $\mathcal{V}$ is said to be a L-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$, if there exist a positive constants $E, F > 0$ such that

$$E\langle Lf, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}, \quad f \in \mathcal{V}. \tag{3.1}$$

The constants E and F are called the bounds of $\{L_i\}_{i\in\mathcal{I}}$. If $E = F \neq 1$, the positive L-operator frame is tight. If $E = F = 1$, it is called a normalized tight L-positive operator frame or a Parseval L-positive operator frame.

Example 3.2. Let $\mathcal{V}$ be a Hilbert C^*-module and $(u_i)_{i\in\mathcal{I}}$ be a L-frame for $\mathcal{V}$, then there exist a positive constants $E, F > 0$ such that

$$E\langle L^* f, L^* f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle f, u_i\rangle_{\mathcal{B}} \langle u_i, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}, \quad f \in \mathcal{V}. \tag{3.2}$$

We define $\{L_i\}_{i\in\mathcal{I}} \subset End_{\mathcal{B}}^*(\mathcal{V})$ by $L_i = u_i \otimes u_i$ for all $i \in \mathcal{I}$.

Then we have,

$$\sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle (u_i \otimes u_i) f, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle \langle f, u_i\rangle_{\mathcal{B}} u_i, f\rangle_{\mathcal{B}} = \sum_{i\in\mathcal{I}} \langle f, u_i\rangle_{\mathcal{B}} \langle u_i, f\rangle_{\mathcal{B}}.$$

Then by (3.2) we have

$$E\langle LL^* f, f\rangle_{\mathcal{B}} = E\langle L^* f, L^* f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}, \quad f \in \mathcal{V}.$$

Let $Q = LL^*$, thus $\{L_i\}_{i\in\mathcal{I}}$ is a Q-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$.

Theorem 3.3. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a L-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$, then $\{L_i\}_{i\in\mathcal{I}}$ corresponds to a L-operator frame. The converse is valid.*

Proof. Let $\{\tilde{L_i}\}_{i\in\mathcal{I}}$ be a $\tilde{L}$-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$.
By the definition of $\tilde{L_i}$ and $\tilde{L}$ then there exist $L_i, R \in End_{\mathcal{B}}^*(\mathcal{V})$ such that $\tilde{L_i} = L_i^* L_i, \quad i \in \mathcal{I}$ and $\tilde{L} = R^* R$.

From the definition of $\{\tilde{L_i}\}_{i\in\mathcal{I}}$ there exist positive constants $E, F > 0$ such that

$$E\langle \tilde{L} f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle \tilde{L_i} f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}; \quad f \in \mathcal{V}.$$

So,

$$E\langle LL^* f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}} \langle L_i^* L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} \quad f \in \mathcal{V}.$$

Thus,

$$E\langle L^*f, L^*f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle L_i f, L_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} \quad f \in \mathcal{V}.$$

Hence, $\{L_i\}_{i\in\mathcal{I}}$ is a L-operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$.

For the converse, let $\{L_i\}_{i\in\mathcal{I}}$ be an L-operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, then there exist positive constants $E, F > 0$ such that

$$E\langle L^*f, L^*f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle L_i f, L_i f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} \quad f \in \mathcal{V},$$

so,

$$E\langle LL^*f, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle L_i^* L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} \quad f \in \mathcal{V},$$

Thus $\{L_i^* L_i\}_{i\in\mathcal{I}}$ is a LL^*-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. □

Corollary 3.4. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a tight L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. Then $\{L_i\}_{i\in\mathcal{I}}$ corresponds to a tight L-operator frame. The converse is also valid.*

Theorem 3.5. *Let $\{\Phi_i \in End^*_{\mathcal{B}}(\mathcal{V}, \mathcal{V}_i) : i \in \mathcal{I}\}$ be a L-g-frame for $\mathcal{V}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$. Then $\{\Phi_i\}_{i\in\mathcal{I}}$ corresponds to an L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. The convers is also valid.*

Proof. Just use characterization for the positive elements in a C^*-algebra. □

Definition 3.6. Let $L \in End^{*^+}_{\mathcal{B}}(\mathcal{V})$ and $\{L_i\}_{i\in\mathcal{I}}$ be a L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. We define the frame operator

$$\begin{aligned} S_L : \mathcal{V} &\longrightarrow \mathcal{V} \\ f &\longmapsto S_L f = \sum_{i\in\mathcal{I}} L_i f \end{aligned}$$

Proposition 3.7. *Let $L \in End^{*^+}_{\mathcal{B}}(\mathcal{V})$ and $\{L_i\}_{i\in\mathcal{I}}$ be a L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$ with bounds E, F and frame operator S_L. Then S_L is positive and self-adjoint. Moreover, we have $EL \leq S \leq FI_{\mathcal{V}}$.*

Proof. It is clear that S_L is positive and self-adjoint.
From the definition of $\{L_i\}_{i\in\mathcal{I}}$, we have,

$$\langle ELf, f\rangle_{\mathcal{B}} = E\langle Lf, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle L_i f, f\rangle_{\mathcal{B}} = \langle S_L f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} = \langle Ff, f\rangle_{\mathcal{B}}.$$

This shows that

$$EL \leq S_L \leq FI_{\mathcal{V}},$$

□

Theorem 3.8. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert C^*-modules over a C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ respectively and let $P \in End^{*^+}_{\mathcal{B}}(\mathcal{V}_1)$ and $Q \in End^{*^+}_{\mathcal{A}}(\mathcal{V}_2)$. Let $\{\Phi_i\}_{i\in\mathcal{I}}$ be a P-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V}_1)$ with bounds E, F and frame operator S_Φ, let $\{\Psi_j\}_{j\in\mathcal{J}}$ be a Q-positive operator frame for $End^*_{\mathcal{A}}(\mathcal{V}_2)$ with bounds M, N and frame operator S_Ψ. Then $\{\Phi_i \otimes \Psi_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a $P \otimes Q$-positive operator frame for $End^*_{\mathcal{B}\otimes\mathcal{A}}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ with frame operator $S_\Phi \otimes S_\Psi$ and bounds EM and FN.*

Proof. By the definition of $\{\Phi_i\}_{i\in\mathcal{I}}$ and $\{\Psi_j\}_{j\in\mathcal{J}}$ we have:

$$E\langle Pf, f\rangle_{\mathcal{B}} \leq \sum_{i\in\mathcal{I}}\langle \Phi_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} \quad f \in \mathcal{V}_1.$$

$$M\langle Qg, g\rangle_{\mathcal{A}} \leq \sum_{j\in\mathcal{J}}\langle \Psi_j g, g\rangle_{\mathcal{A}} \leq N\langle g, g\rangle_{\mathcal{A}} \quad g \in \mathcal{V}_2.$$

Therefore,

$$EM\langle Pf, f\rangle_{\mathcal{B}} \otimes \langle Qg, g\rangle_{\mathcal{A}} \leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \sum_{j\in\mathcal{J}} \langle \Psi_j g, g\rangle_{\mathcal{A}} \leq FN\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}}.$$

Then,

$$EM\langle Pf \otimes Qg, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \langle \Psi_j g, g\rangle_{\mathcal{A}} \leq FN\langle f\otimes g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}.$$

Consequently, we have for all $f\otimes g \in \mathcal{V}_1 \otimes \mathcal{V}_2$

$$EM\langle (P\otimes Q)(f\otimes g), f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f \otimes \Psi_j g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \leq FN\langle f\otimes g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}.$$

Then,

$$EM\langle (P\otimes Q)(f\otimes g), f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle (\Phi_i \otimes \Psi_j)(f\otimes g), f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}} \leq FN\langle f\otimes g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}.$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{V}_1 \otimes_{alg} \mathcal{V}_2$ and then it's satisfied for all $h \in \mathcal{V}_1 \otimes \mathcal{V}_2$. It shows that $\{\Phi_i \otimes \Psi_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a $P\otimes Q$-positive operator frame for $End^*_{\mathcal{B}\otimes\mathcal{A}}(\mathcal{V}_1\otimes\mathcal{V}_2)$ with bounds EM and FN.
By the definition of frame operator S_Φ and S_Ψ we have:

$$S_\Phi f = \sum_{i\in\mathcal{I}} \Phi_i f \quad f \in \mathcal{V}_1.$$

$$S_\Psi g = \sum_{j\in\mathcal{J}} \Psi_j g \quad g \in \mathcal{V}_2.$$

Therefore:

$$\begin{aligned}(S_\Phi \otimes S_\Psi)(f\otimes g) &= S_\Phi f \otimes S_\Psi g \\ &= \sum_{i\in\mathcal{I}} \Phi_i f \otimes \sum_{j\in\mathcal{J}} \Psi_j g \\ &= \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \Phi_i f \otimes \Psi_j g \\ &= \sum_{i\in\mathcal{I}, j\in\mathcal{J}} (\Phi_i \otimes \Psi_j)(f\otimes g).\end{aligned}$$

By the uniqueness of frame operator, the last expression is equal to $S_{\Phi\otimes\Psi}(f\otimes g)$. Consequently we have $(S_\Phi \otimes S_\Psi)(f\otimes g) = S_{\Phi\otimes\Psi}(f\otimes g)$. The last equality is satisfied for every finite sum of elements in $\mathcal{V}_1 \otimes_{alg} \mathcal{V}_2$ and then it's satisfied for all $h \in \mathcal{V}_1\otimes\mathcal{V}_2$. It shows that $(S_\Phi \otimes S_\Psi)(h) = S_{\Phi\otimes\Psi}(h)$, so $S_{\Phi\otimes\Psi} = S_\Phi \otimes S_\Psi$. □

4 $*$-positive Operator Frame

Definition 4.1. Let $\mathcal{V}$ be a Hilbert C^*-module over a unitary C^*-algebra $\mathcal{B}$. A sequence $\{L_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ is said to be a $*$-positive operator frame for $\mathcal{V}$ if there exist strictly nonzero elements E, F in $\mathcal{B}$ such that

$$E\langle f, f\rangle_{\mathcal{B}} E^* \leq \sum_{i\in\mathcal{I}} \langle L_i f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}} F^*; \quad f \in \mathcal{V}. \tag{4.1}$$

The elements E and F are called the bounds of $\{L_i\}_{i\in\mathcal{I}}$.
If $E = F = 1_{\mathcal{B}}$, it is called a normalized tight $*$-positive operator frame.

Proposition 4.2. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. Then $\{L_i\}_{i\in\mathcal{I}}$ corresponds to a $*$-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$. The convere is also valid.*

Proof. $\Longrightarrow$) Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$, then there exist strictly nonzero elements E, F such that

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_if, L_if\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^*; \quad f\in\mathcal{V}, \tag{4.2}$$

so, we have,

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_i^*L_if, f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}; F^* \quad f\in\mathcal{V}, \tag{4.3}$$

by setting $R_i = L_i^*L_i \quad i\in\mathcal{I}$, we obtain

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_if, f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^* \quad f\in\mathcal{V}, \tag{4.4}$$

then $\{R_i\}_{i\in\mathcal{I}} \subset End_{\mathcal{B}}^*(\mathcal{V})$ is a $*$-positive operator frame for $\mathcal{V}$.
$\Longleftarrow$) Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$, then there exist strictly nonzero element E, F such that

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_if, f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^* \quad f\in\mathcal{V}, \tag{4.5}$$

since L_i is a positive operator then $L_i = R_i^*R_i \quad i\in\mathcal{I}$, from last inequality we have

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_i^*R_if, f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^* \quad f\in\mathcal{V}. \tag{4.6}$$

So,

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_if, R_if\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^* \quad f\in\mathcal{V}, \tag{4.7}$$

which give the sequence $\{R_i\}_{i\in\mathcal{I}} \subset End_{\mathcal{B}}^*(\mathcal{V})$ is a $*$-positive operators frame for $\mathcal{V}$ with bounds E and F. □

Let $\{L_i\}_{i\in\mathcal{I}} \subset End_{\mathcal{B}}^*(\mathcal{V})$ be a $*$-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V})$ with bounds E and F.
We define the frame operator associated with $\{L_i\}_{i\in\mathcal{I}}$ by

$$S_L : \mathcal{V} \longrightarrow \mathcal{V}$$
$$f \longmapsto S_Lf = \sum_{i\in\mathcal{I}} L_if.$$

Proposition 4.3. *The frame operator S_L is self adjoint, positive and invertible and we have*

$$\|E^{-1}\|^{-2} \leq \|S\| \leq \|F\|^2.$$

Proof. Results from Lemma 1.2 □

Theorem 4.4. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert C^*-modules over unitary C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ respectively. Let $\{\Phi_i\}_{i\in\mathcal{I}}$ be a $*$-positive operator frame for $End_{\mathcal{B}}^*(\mathcal{V}_1)$ with bounds E, F and frame operator S_Φ. Let $\{\Psi_j\}_{j\in\mathcal{J}}$ be a $*$-positive operator frame for $End_{\mathcal{A}}^*(\mathcal{V}_2)$ with bounds M, N and frame operator S_Ψ. Then the sequence $\{\Phi_i \otimes \Psi_i\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a $*$-positive operator frames for $End_{\mathcal{B}\otimes\mathcal{A}}^*(\mathcal{V}_1 \otimes \mathcal{V}_2)$ with bounds $E\otimes M$ and $F\otimes N$ and frame operator $S_{\Phi\otimes\Psi} = S_\Phi \otimes S_\Psi$.*

Proof. By the definition of $\{\Phi_i\}_{i\in\mathcal{I}}$ and $\{\Psi_j\}_{j\in\mathcal{J}}$, we have

$$E\langle f,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle \Phi_i f,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^* \qquad f\in\mathcal{V}_1,$$

and

$$M\langle g,g\rangle_{\mathcal{A}}M^* \leq \sum_{j\in\mathcal{J}}\langle \Psi_i g,g\rangle_{\mathcal{A}} \leq N\langle g,g\rangle_{\mathcal{A}}N^* \qquad g\in\mathcal{V}_2,$$

therefore

$$\begin{aligned}&(E\langle f,f\rangle_{\mathcal{B}}E^*)\otimes(M\langle g,g\rangle_{\mathcal{A}}M^*)\\ &\leq \left(\sum_{i\in\mathcal{I}}\langle \Phi_i f,f\rangle_{\mathcal{B}}\right)\otimes\left(\sum_{j\in\mathcal{J}}\langle \Psi_j g,g\rangle_{\mathcal{A}}\right)\\ &\leq (F\langle f,f\rangle_{\mathcal{B}}F^*)\otimes(N\langle g,g\rangle_{\mathcal{A}}N^*)\end{aligned}$$

then, we have :

$$\begin{aligned}&(E\otimes M)(\langle f,f\rangle_{\mathcal{B}}\otimes\langle g,g\rangle_{\mathcal{A}})((E^*\otimes M^*)\\ &\leq \sum_{i\in\mathcal{I},j\in\mathcal{J}}\langle \Phi_i f,f\rangle_{\mathcal{B}}\otimes\langle \Psi_j g,g\rangle_{\mathcal{A}}\\ &\leq (F\otimes N)(\langle f,f\rangle_{\mathcal{B}}\otimes\langle g,g\rangle_{\mathcal{A}})(F^*\otimes N^*)\end{aligned}$$

so,

$$\begin{aligned}&(E\otimes M)(\langle f\otimes g,f\otimes g\rangle_{\mathcal{A}\otimes\mathcal{B}})((E^*\otimes F^*)\\ &\leq \sum_{i\in\mathcal{I},j\in\mathcal{J}}\langle \Phi_i f\otimes\Psi_j g,f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}\\ &\leq (F\otimes N)(\langle f,f\rangle_{\mathcal{B}}\otimes\langle g,g\rangle_{\mathcal{A}})((F^*\otimes N^*)\end{aligned}$$

then

$$\begin{aligned}&(E\otimes M)(\langle f\otimes g,f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}})(E^*\otimes M^*)\\ &\leq \sum_{i\in\mathcal{I},j\in\mathcal{J}}\langle \Phi_i f\otimes\Psi_j g,f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}\\ &\leq (F\otimes N)(\langle f,f\rangle_{\mathcal{B}}\otimes\langle g,g\rangle_{\mathcal{A}})(F^*\otimes N^*)\end{aligned}$$

which give,

$$\begin{aligned}&(E\otimes M)(\langle f\otimes g,f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}})(E\otimes M)^*\\ &\leq \sum_{i\in\mathcal{I},j\in\mathcal{J}}\langle (\Phi_i\otimes\Psi_j)(f\otimes g),f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}\\ &\leq (F\otimes N)(\langle f\otimes g,f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}})(F\otimes N)^*.\end{aligned}$$

The last inequality is satisfied for every finite elements in $\mathcal{V}_1\otimes\mathcal{V}_2$ and then it is satisfied for all $h\in\mathcal{V}_1\otimes\mathcal{V}_2$. It show that $\{\Phi_i\otimes\Psi_j\}_{i\in\mathcal{I},j\in\mathcal{J}}$ is a $*$-positive operator frames for $End^*_{\mathcal{B}\otimes\mathcal{A}}(\mathcal{V}_1\otimes\mathcal{V}_2)$ with bounds $E\otimes M$ and $F\otimes N$. □

5 $*$-L-positive Opertor Frame

Definition 5.1. Let $L\in End^{*^+}_{\mathcal{B}}(\mathcal{V})$ and $\{L_i\}_{i\in\mathcal{I}}$ be a sequence of adjointable operators for a Hilbert C^*-module $\mathcal{V}$. Then $\{L_i\}_{i\in\mathcal{I}}$ is called a $*$-L-positive operator frame for $\mathcal{V}$ if there exist strictly nonzero elements E,F in $\mathcal{B}$ such that

$$E\langle Lf,f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_i f,f\rangle_{\mathcal{B}} \leq F\langle f,f\rangle_{\mathcal{B}}F^*; \quad f\in\mathcal{V}. \tag{5.1}$$

The elements E and F are called the bounds of $\{L_i\}_{i\in\mathcal{I}}$.
If $E = F = 1$, it is called a normalized tight $*$-L-positive operator frame.

Proposition 5.2. *Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-L-operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. Then $\{L_i\}_{i\in\mathcal{I}}$ corresponds to a $*$-L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$. The convere is also valid.*

Proof. $\Longrightarrow$) Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-L-operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, then there exist strictly nonzero elements E, F such that

$$E\langle L^*f, L^*f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_if, L_if\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^*; \quad f \in \mathcal{V}. \tag{5.2}$$

So, we have,

$$E\langle LL^*f, f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_i^*L_if, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^*; \quad f \in \mathcal{V}, \tag{5.3}$$

by setting $R_i = L_i^*L_i$ and $Q = LL^*$ we obtain

$$E\langle Qf, f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_if, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^* \quad f \in \mathcal{V}. \tag{5.4}$$

Then, $\{R_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ is a $*$-Q-positive operator frame for $\mathcal{V}$.
$\Longleftarrow$) Let $\{L_i\}_{i\in\mathcal{I}}$ be a $*$-L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$, then there exist strictly nonzero element E, F such that

$$E\langle Lf, f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle L_if, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^* \quad f \in \mathcal{V}. \tag{5.5}$$

Since L_i and L are a positive operators for $End^*_{\mathcal{B}}(\mathcal{V})$, then $L_i = R_i^*R_i, \quad i \in \mathcal{I}$ and $L = QQ^*$. From the last inequality we have

$$E\langle QQ^*f, f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_i^*R_if, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^* \quad f \in \mathcal{V}. \tag{5.6}$$

So,

$$E\langle Q^*f, Q^*f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle R_if, R_if\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^* \quad f \in \mathcal{V}, \tag{5.7}$$

which give the sequence $\{R_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ is a $*$-Q-positive operator frame for $\mathcal{V}$ with bounds E and F. □

Let $\{L_i\}_{i\in\mathcal{I}} \subset End^*_{\mathcal{B}}(\mathcal{V})$ be a $*$-L-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V})$ with bounds E and F.
We define the frame operator associated with $\{L_i\}_{i\in\mathcal{I}}$ by

$$S_L : \mathcal{V} \longrightarrow \mathcal{V}$$
$$f \longmapsto S_Lf = \sum_{i\in\mathcal{I}} L_if$$

Remark 5.3. By the definition, the frame operator S_L is self adjoint.

Theorem 5.4. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Hilbert C^*-modules over unitary C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ respectively and let $P \in End^*_{\mathcal{B}}(\mathcal{V}_1)$ and $Q \in End^*_{\mathcal{A}}(\mathcal{V}_2)$. Let $\{\Phi_i\}_{i\in\mathcal{I}}$ be a $*$-P-positive operator frame for $End^*_{\mathcal{B}}(\mathcal{V}_1)$ with bounds E, F and frame operator S_Φ and let $\{\Psi_j\}_{j\in\mathcal{J}}$ be a $*$-Q-positive operator frame for $End^*_{\mathcal{A}}(\mathcal{V}_2)$ with bounds M, N. Then the sequence $\{\Phi_i\otimes\Psi_j\}_{i\in\mathcal{I},j\in\mathcal{J}}$ is a $*$-$(P\otimes Q)$-positive operator frames for $End^*_{\mathcal{B}\otimes\mathcal{A}}(\mathcal{V}_1\otimes\mathcal{V}_2)$ with bounds $E\otimes M$ and $F\otimes N$ and the frame operator $S_{\Phi\otimes\Psi} = S_\Phi \otimes S_\Psi$.*

Proof. By the definition of $\{\Phi_i\}_{i\in\mathcal{I}}$ and $\{\Psi_j\}_{j\in\mathcal{J}}$ we have

$$E\langle P^*f, P^*f\rangle_{\mathcal{B}}E^* \leq \sum_{i\in\mathcal{I}}\langle \Phi f, f\rangle_{\mathcal{B}} \leq F\langle f, f\rangle_{\mathcal{B}}F^*; \qquad f \in \mathcal{V}_1,$$

and

$$M\langle Q^*g, Q^*g\rangle_{\mathcal{A}} M^* \leq \sum_{j\in\mathcal{J}} \langle \Psi_i g, g\rangle_{\mathcal{A}} \leq N\langle g, g\rangle_{\mathcal{A}} N^*; \qquad g \in \mathcal{V}_2.$$

Therefore

$$\begin{aligned}(E\langle P^*f, P^*f\rangle_{\mathcal{B}} E^*) &\otimes (M\langle Q^*g, Q^*g\rangle_{\mathcal{A}} M^*)\\ &\leq \sum_{i\in\mathcal{I}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \sum_{j\in\mathcal{J}} \langle \Psi_j g, g\rangle_{\mathcal{A}}\\ &\leq (F\langle f, f\rangle_{\mathcal{B}} F^*) \otimes (N\langle g, g\rangle_{\mathcal{A}} N^*).\end{aligned}$$

Then we have

$$\begin{aligned}(E\otimes M)(\langle P^*f, P^*f\rangle_{\mathcal{B}} &\otimes \langle Q^*g, Q^*g\rangle_{\mathcal{A}})((E^*\otimes M^*)\\ &\leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f, f\rangle_{\mathcal{B}} \otimes \langle \Psi_j g, g\rangle_{\mathcal{A}}\\ &\leq (F\otimes N)(\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}})(F^*\otimes N^*).\end{aligned}$$

So,

$$\begin{aligned}(E\otimes M)(\langle P^*f \otimes Q^*g, P^*f \otimes Q^*g\rangle_{\mathcal{B}\otimes\mathcal{A}})((E^*\otimes M^*)\\ \leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f \otimes \Psi_j g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}\\ \leq (F\otimes N)(\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}})(F^*\otimes N^*).\end{aligned}$$

Then

$$\begin{aligned}(E\otimes M)(\langle (P\otimes Q)^*(f\otimes g), (P\otimes Q)^*(f\otimes g)\rangle_{\mathcal{A}\otimes\mathcal{B}})(E\otimes M)^*\\ \leq \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle \Phi_i f \otimes \Psi_j g, f\otimes g\rangle_{\mathcal{B}\otimes\mathcal{A}}\\ \leq (F\otimes M)(\langle f, f\rangle_{\mathcal{B}} \otimes \langle g, g\rangle_{\mathcal{A}})(F\otimes N)^*.\end{aligned}$$

The last inequality is satisfied for every finite elements in $\mathcal{V}_1 \otimes \mathcal{V}_2$ and then it is satisfied for all $h \in \mathcal{V}_1 \otimes \mathcal{V}_2$. It show that $\{\Phi_i \otimes \Psi_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a $*$-$(P\otimes Q)$-positive operator frames for $End^*_{\mathcal{B}\otimes\mathcal{A}}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ with bounds $E\otimes M$ and $F\otimes N$.

□

References

[1] Alijani, A., and Dehghan, M., 2011, $*$-Frames in Hilbert $\mathcal{C}^*$-modules, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 73(4), 89-106.

[2] Arambašić, L., 2007, On frames for countably generated Hilbert $\mathcal{C}^*$-modules, *Proc. Am. Math. Soc.* 135, 469-478.

[3] Asgari, M. S., and Rahimi H., 2014, Generalized frames for operators in Hilbert spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 17(2), 1450013.

[4] Conway, J. B., 2000, *A course in operator theory*, American Mathematical Society, 21.

[5] Daubechies, I., Grossmann, A., and Meyer, Y., 1986, Painless nonorthogonal expansions, *J. Math. Phys.*, 27, 1271-1283.

[6] Davidson, F. R., 1996, *$\mathcal{C}^*$-algebra by example*, Fields Institute monographes, Monog.

[7] Duffin, R. J., and Schaeffer A. C., 1952, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* 72, 341-366.

[8] Frank, M., and Larson D. R., 2002, Frames in Hilbert $\mathcal{C}^*$-modules and $\mathcal{C}^*$-algebras, *J. Oper. Theory*, 48, 273-314.

[9] Gabor, D., 1946, Theory of communications, *J. Elec. Eng.*, 93, 429-457.

[10] Kabbaj, S., and Rossafi, M., 2018, $*$-operator frame for $End_{\mathcal{A}}^*(H)$, *Wavelet Linear Algebra*, 5(2), 1-13.

[11] Li, C. Y., and Cao, H. X., 2006, Operator frames for $B(H)$: Wavelet analysis and applications, Tao Qian, Mang I. Vai and Xu Yuesheng (eds.), *Applied and Numerical Harmonic Analysis*, 67-82.

[12] Paschke, W., 1973, Inner product modules over B^*-algebras, *Trans. Amer. Math. Soc.*, 182, 443-468.

[13] Rossafi, M., and Kabbaj, S., 2020, $*$-K-operator frame for $End_{\mathcal{A}}^*(H)$, *Asian-Eur. J. Math*, 13, 2050060.

[14] Rossafi, M., and Bourouihiya A., Labrigui H., and Touri, A., 2020, The duals of $*$-operator frame for $End_{\mathcal{A}}^*(H)$, *Asia Math.*, 4, 45-52.

[15] Rossafi, M., Touri, A., Labrigui, H., and Akhlidj, A., 2019, Continuous $*$-K-g-Frame in Hilbert C^*-Modules, *Journal of Function Spaces*, Article ID 2426978.

[16] Rossafi, M., and Kabbaj, S., 2019, Operator frame for $End_{\mathcal{A}}^*(H)$, *J. Linear Topol. Algebra*, 8, 85-95.

[17] Rossafi, M., and Kabbaj, S., 2018, $*$-K-g-frames in Hilbert $\mathcal{A}$-modules, *J. Linear Topol. Algebra*, 7, 63-71.

[18] Rossafi, M., and Kabbaj, S., 2018, $*$-g-frames in tensor products of Hilbert C^*-modules, *Ann. Univ. Paedagog. Crac. Stud. Math.*, 17, 17-25.

[19] Rossafi, M., and Kabbaj, S., 2018, K-operator frame for $End_{\mathcal{A}}^*(H)$, *Asia Math.*, 2, 52-60.

[20] Shekhar, C., and Kaushik, S. K., 2017, Frames for $B(H)$, *Oam*, 11(1), 181-198.

[21] Sun, W., 2006, G-frames and g-Riesz bases, *J. Math. Anal. Appl.*, 322(1), 437-452.