Some Fundamental Research Tools in Mathematical Sciences

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Abstract: *Research in mathematical sciences either builds up the insight or breaks the boundary of the literature of the pertaining mathematical research area. A mathematical research technique is an attentive, persistent and systematic approach based on the logical rules of inference and mathematical rules of inference to find something new. Mathematical modeling, construction of theorems with the proofs, design of algorithms, data with simulation could be considered as the fundamental tools in mathematical research. In this paper, we discuss some fundamental research tools which are useful to do research in mathematical sciences.*

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1 Introduction

Research in mathematical sciences contributes either for the development of mathematical structure or for the implementation of the developed mathematical structure in order to build insight up or contribute in breaking the boundary of the existing literature of one or more aspects of mathematical sciences. Objective of the research would be one or more among the discovery or invention of new facts, the verification of proposed or established facts in the specified research issue, the analysis of proposed facts to identify the cause and effect relationships, the development of existing theory, the solution of the proposed problem, and so on [16]. There exist different approaches for research, for example, theoretical, experimental, empirical *etc.* However, the research we basically discuss in this paper is theoretical. The structure of research in mathematical sciences can be observed in [10]. The theoretical research is foremost important in mathematical sciences for the structural development of various mathematical approaches and also implementation of such developed structures in various mathematically related fields such as physical or biological sciences. The research in mathematics is usually conducted by the list of valid arguments. Usually a statement as a conjecture is set up which is either proved by the list of valid arguments or is refuted by an example which we call a counter-example. The language of research in mathematical sciences is based on two inferences. One is the logical rule of inference which derives a new fact through the valid arguments and the other is mathematical rules of inference which derives a new fact through the valid mathematical properties under given circumstances. Basically there are four major tools in research in mathematical sciences, namely, mathematical modeling, development of mathematical theorems, design of algorithms and data handling. In this paper, we briefly discuss the aforementioned four research tools.

The paper is organized as follows. Section 2 briefly discusses on mathematical formulation, Section 3 is for how theorems are developed, Section 4 discusses about the algorithm, Section 5 discuss about simulation and the last section concludes the paper.

2 Mathematical modeling

Mathematical modeling is an abstraction or simplification of a real world problem to identify the behavior of the system [11]. In modeling, a space which incorporates all the structures of the real world problem is considered. The structures are parameterized and relationships among the parameters are investigated during the modeling process. If the variables or the parameters follow the discrete or the continuous structures, the model formed becomes discrete or the continuous mathematical models, accordingly. If the state variables, the variables which change the state, change at a countable stage of time, the structure is discrete. Moreover, if instead, the state variables change at any instance of time, the structure is continuous. Modeling a real phenomenon into a mathematical form is always a difficult task because it involves the translation/conversion of imprecise assumptions into very precise mathematical formulation/formulas. The challenge in modeling is not to develop the most comprehensive one but possible simple that incorporates major structures of the problem [1]. Validation of the model is another challenge. If the model does not validate the real problem, the model is to get updated which leads the model to be less precise [11]. The models may be linear or non-linear according to less complex to more complex relationships among the structures of the problem. The models with less complex relationships among the structures demand analytical solution. However, the models with more complex relationships demand numerical solutions. The mathematical models can be developed by using any mathematical approaches for example algebraic, geometric, number theoretic, topological, analytical, graph theoretic, *etc.* which depend on the nature of the space and the interrelationships among the variables of the structures of the problems, see [1, 9, 11, 18]. One may see how a real world problem can be modeled using graph theoretic approach in [3]. In mathematical sciences research either we adopt already developed model or modify the developed model or develop new model according to the need of the research. After we decide the modeling step, we validates the model through the development of the insightful theorems and the mathematical properties.

3 Theorems

If the existing model is modified or a new model is developed, such a model is to be validated. Likewise if new properties of the research problem are investigated, their truth has to be assured through theorems. Theorem is an important and widely used research tool in the research of mathematical sciences. A theorem is a statement with the proof. Statement follows the theory of excluded middle which means the statement is either true or false. The statement may be conditional like if p, then q or bi-conditional like p if and only if q. In conditional statement, $p \Rightarrow q$, p is called hypothesis and q is conclusion and p and q are hypothesis and conclusion both in bi-conditional statement $p \Leftrightarrow q$. Proof follows logical rules of inference together with mathematical rules of inference through a list of valid arguments. The list of valid arguments consists of premises followed by a conclusion. An argument is valid if premises are true then the conclusion is also true [7]. A statement is a conjecture until it is either proved or is refuted with a counterexample, see [15]. The logical rules of inference help to construct a proof of a statement. There are several such rules for example, Modus Ponens and Modus Tollens. However, there may be fallacy which does not prove that the statement is true. Converse error and inverse error are two examples of fallacies, see [7]. There have been a few methods for the proof. The direct proof for "if p, then q" is that the hypothesis p is assumed to be true and the conclusion q is assured to be true through a series of valid arguments which depend on the logical and the mathematical rules of inferences. If $\neg q$, then $\neg p$ is the contrapositive approach. If p is supposed to be not true, then one reaches a contradiction with a valid argument approach of proof is called contradiction method, see [4]. Mathematical induction assures a countable infinity of statements true in a finite number of steps [5]. There are terminology similar to a theorem. A lemma is a theorem as an aid to a theorem whereas a corollary is also a theorem as a consequence of a theorem. A conjecture is a statement which is open that means it has been neither proved nor refuted. Designing a conjecture is important to a researcher [15].

4 Algorithms

The problem designed in mathematical formulation is solved using appropriate algorithms. If there exists no such an algorithm, the researcher is supposed to investigate it. Algorithm is an useful research tools in mathematical sciences. An algorithm is a mathematical technique which can be used to solve each instance of a given problem [14]. An algorithm incorporates finiteness, effectiveness, termination, determinism, correctness as its properties. Moreover, it is expected to be efficient. The algorithm consists of an input which is an instance of a problem, a concise and a precise pseudocode which yields a solution and an output, a solution to the problem. The pseudocode is written as a list of block structures, for example "if else" indentation. The pseudocode may consists of a ramification, a loop, an iteration, and so on. The ramification "If b then $p_1, p_2, \ldots, p_n$ else $q_1, q_2, \ldots, q_m$" assures, if b is true, then $p_1, p_2, \ldots, p_n$, are executed and if b is not true, then $q_1, q_2, \ldots, q_m$, are executed. Likewise, loop "for $i = 1, 2, \ldots, n$ do $p_1, p_2, \ldots, p_n$" means all operations $p_1, p_2, \ldots, p_n$, are executed for each $i = 1, 2, \ldots, n$. Iteration "while b do $p_1, p_2, \ldots, p_n$" is recursive that $p_1, p_2, \ldots, p_n$ are executed as long as b holds. Most of algorithms are recursive [6].

There are different types of algorithms based on solution. An exact algorithm yields optimal solution though the algorithm may be expensive in terms of time and space. Approximate algorithm yields near to optimal solution. Moreover, heuristic one seeks optimal solution with no guarantee [19]. Algorithms can be based on the concept also. For example, divide and conquer algorithms divide the problem into smaller subproblems of the same type and solve them recursively. Ultimate solution to the original problem is obtained by combining them. Dynamic programming algorithms remember past results and uses them to find new results. Greedy algorithms consider the best right now and ignores the future [6]. The performance of algorithm is another important notion. Researchers in mathematical sciences basically consider the worst case analysis of the performance in which even pathological instances are considered. The worst case complexity incorporates space complexity which concerns the memory and the time complexity which is expressed as a function of computer steps required to execute an algorithm. An algorithm is said to run in $O(f(n))$ time if for some numbers c and n_o, the time taken by the algorithm is at most $cf(n)$ for all $n \geq n_o$ [8]. Problem classes can also be defined based on how algorithms run. A problem is said to be in the problem class P if there exists an algorithm, which runs in polynomial time, as a solution procedure to the problem. A problem is in NP if yes-solution can be verified with a polynomial time algorithm. A problem class is NP-complete if it is NP and all the remaining problems in the class can also be reducible to one another [8].

5 Simulation

In many cases, the real field scenario cannot be replicated in laboratory experiments. A large-scaled experiments demand high cost, and the computer based simulations are cost effective and do not harm the nature. This reflects the importance of simulation in research. An important and widely used research tool in mathematical sciences research is simulation. Simulation is a scientific computing technique which is used in gaining insights into the operation of a system, developing operating or resource policies to improve system performance, testing new concepts and/or systems before implementation or gaining information without disturbing the actual system, see [2]. It explains the behavior of a system and is used in situations where it is difficult to solve problems analytically. A real world problem is formulated as a mathematical model and is planned to fit appropriate data for simulation. Simulation undergoes through verification and validation. Simulation has been replacing experiments leading to insights [13]. Analytical solutions to many mathematical models are inadequate and are limited. Such models are solved through numerical approximations. Correctness and accuracy are important for good numerical approximations which are assured through verification. Verification consists of code verification for computer code and solution verification for estimation of numerical error [17]. Verification is associated with the validation for model quality. Validation means the measurement of the degree of accuracy of a computerized model with the real world situation. Greater the accuracy is more valid the model is, see [2]. The validation in mathematical sciences is usually quantitative which means the direct comparison of the computerized model with the target system. There may be different approaches for such comparison [11, 12]. There may be sources of error and uncertainty because of the randomness existing in the system [2].

6 Conclusion

In this paper, we have briefly discussed some research tools which have been widely used in the research of mathematical sciences. Research in mathematical sciences either develops mathematical structures or implements the developed mathematical structures. The development of mathematical structures and their implementation are based on the logical rules of inferences and mathematical rules of inferences. Usually, research in mathematical sciences are not based on empirical experiments rather on the modeling mathematical formulation, solution techniques and interpretation of the solutions. The mathematical modeling, design of theorems and their proof techniques, design of algorithm and simulations are the research tools in the research of mathematical sciences. Further investigation on the research tools and their recent impacts would be the future research work.

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