Dynamics of Almost Abelian Transcendental Semigroups

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Abstract: *In this article, we show that the escaping, Julia, and Fatou sets of an almost abelian transcendental semigroup of finite type coincide with escaping, Julia and Fatou sets of their respective cyclic subsemigroups*

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1 Introduction

Let $\mathbb{C}$ be the set of all the complex numbers. Let us define a set

$$\mathscr{G} = \{f_\alpha : f_\alpha \text{ is a complex analytic function defined on } \mathbb{C} \; \forall \; \alpha \in \Delta\},$$

where Δ is an index set. Let us also define a set $S = \langle f_\alpha \rangle$ which is called a complex semigroup generated by the elements of $\mathscr{G}$. S is called a rational semigroup if $\mathscr{G}$ is a set of rational functions, and it is called a transcendental semigroup if $\mathscr{G}$ is a set of transcendental entire functions. S is said to be an *abelian semigroup* if its generators commute. If $S = \langle f_1, f_2, \ldots, f_n \rangle$, then it is said to be a *finitely generated semigroup*. If $S = \langle f \rangle$, then S is said to be a *cyclic semigroup*.

If every sequence (f_α) in $\mathscr{G}$ has uniformly converging subsequence (f_{α_k}) on all compact subsets of $\mathbb{C}$, then $\mathscr{G}$ is called a *normal family*. $\mathscr{G}$ is normal at $z \in \mathbb{C}$, if $\mathscr{G}$ is a normal family in any neighborhood U of z. The *Fatou set* $F(S)$ of a semigroup S is a maximal open set where S is a normal family. The complement of $F(S)$ in $\mathbb{C}$ is called the *Julia set* $J(S)$. Also, the set defined by

$$I(S) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \; \forall \; f \in S\}$$

is called an *escaping set* of a semigroup S. If $S = \langle f \rangle$, then $I(f), J(f)$ and $F(f)$, respectively, denote the escaping, Julia, and Fatou sets. Note that these are escaping, Julia, and Fatou sets of a single function f, and for the structure and properties of these sets, and for further dynamical study, we refer to [1, 2, 6, 11].

The study of the Julia and Fatou sets of rational semigroups were initiated by Hinkannen and Martin [3, 4, 5], and that of transcendental semigroups by Poon [12], Haung [7], and others. Kumar and Kumar [8, 9, 10] studied escaping sets of transcendental semigroups. We also studied some of the structure and properties of these sets of rational or transcendental semigroup in the papers [14, 15, 16, 17, 18]. Here, we study these sets of an almost abelian transcendental semigroup.

Definition 1.1. Let $\Psi = \{\psi : \psi(z) = cz + d \; \forall \; c, d \in \mathbb{C} \text{ with } c \neq 0\}$, and let S be a transcendental semigroup. We say that S is *almost abelian* if

1. $\psi(F(S)) = F(S) \; \forall \; \psi \in \Psi$,
2. there exists $\psi \in \Psi$ such that $\psi \circ g \circ f = f \circ g \; \forall \; f, g \in S$.

In this article, we prove the assertion given below.

Proposition 1.1. *Let S be an almost abelian transcendental semigroup of finite type. Then for each $g \in S$, we have $I(S) = I(g)$, $J(S) = J(g)$ and $F(S) = F(g)$.*

Note that we define and discuss a transcendental semigroup of finite type in Section 3.

2 Almost Abelian Transcendental Semigroups

Definition 1.1 of almost abelian transcendental semigroup looks more restrictive on an affine map $\psi(z) = cz+d,\ c \neq 0$, and this type of function can play the role of semi-conjugacy. We say that a function f is (semi) conjugated to another function g if there is a continuous function ψ such that $\psi \circ f = g \circ \psi$. For example, $f(z) = \gamma \cos z,\ \gamma \in \mathbb{C}$ is semi-conjugated to $g(z) = -\gamma \cos z$ because there is a function $\psi(z) = -z$ such that $\psi \circ f = g \circ \psi$. If there is a complex semigroup generated by such type of semi-conjugate transcendental entire functions, then the semigroup will more likely be almost abelian.

Theorem 2.1. *Let $S = \langle f_1, f_2, f_3, \ldots \rangle$ be a transcendental semigroup, and ψ be a function $z \to cz + d\ \forall\ c,\ d \in \mathbb{C}$ with $c \neq 0$ such that $\psi \circ f_i = f_i \circ \psi\ \forall\ i$, and $f_j = \psi \circ f_i\ \forall\ i \neq j$. Then the semigroup S is almost abelian.*

Lemma 2.1. *Let $S = \langle f_1, f_2, f_3 \ldots \rangle$ be a transcendental semigroup, and ψ be function $z \to cz+d\ \forall\ c,\ d \in \mathbb{C}$ with $c \neq 0$. If $\psi \circ f_i = f_j \circ \psi\ \forall\ i \neq j$. Then $\psi(F(S)) = F(S)$ and $\psi(J(S)) = J(S)$.*

Proof. First, we show that if $\psi \circ f_i = f_j \circ \psi\ \forall\ i \neq j$, then $\psi \circ f = g \circ \psi\ \forall\ f,\ g \in S$. By definition, for all $f,\ g \in S$, we have $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $g = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$. Now, $\psi \circ f = \psi \circ f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} = f_{j_1} \circ \psi \circ \circ f_{i_2} \circ \ldots \circ f_{i_n} = \ldots = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n} \circ \psi = g \circ \psi$. This proves our claim.

If $w \in \psi(F(S))$, then there is $z_0 \in F(S)$ such that $w = \psi(z_0)$. Take a neighborhood $U \subset F(S)$ of z_0 with $|f(z) - f(z_0)| < \epsilon/2\ \forall\ z \in U$ and $f \in S$. This shows that $f(U)$ has a diameter less than ϵ for all $f \in S$. The function ψ has the bounded first derivative $a \neq 0$, so it is a Lipschitz with Lipschitz constant $k = \sup|\psi^{'}(z)| = a$. Now, for any $g \in S$, the diameter of $g(\psi(U)) = \psi(f(U))$ is less than $k\epsilon$. Hence $w = \phi(z_0) \in F(S)$. This proves $\psi(F(S)) \subset F(S)$.

Next, let $w \in \psi(J(S))$. Then $w = \psi(z_0)$ for some $z_0 \in J(S)$. Let z_0 be a repelling fixed point for some $f \in S$, but not a critical point of ψ. Then $\psi \circ f = g \circ \psi \Longrightarrow g$ has a fixed point at $\psi(z_0)$ with the same multiplier as that of f at z_0. Thus, ψ maps repelling fixed points of any $f \in S$ to repelling fixed points of another $g \in S$. By [12, Theorem 4.1 and 4.2], $J(S)$ is perfect and

$$J(S) = \overline{\bigcup_{f \in S} J(f))},$$

Therefore, it follows that $\psi(J(S)) \subset J(S)$.

Finally, the fact $\psi(\mathbb{C}) = \mathbb{C}$ is obvious. Using this fact in $F(S) = \mathbb{C} - J(S)$ and $J(S) = \mathbb{C} - F(S)$, we get

$$\psi(F(S)) = \mathbb{C} - \psi(J(S)) \text{ and } \psi(J(S)) = \mathbb{C} - \psi(F(S)). \tag{2.1}$$

Again, using facts $\psi(J(S)) \subset J(S)$ and $\psi(F(S)) \subset F(S)$ in (2.1), we will get required reverse inclusions $F(S) \subset \psi(F(S))$ and $J(S) \subset \psi(J(S))$. □

Note that Lemma 2.1 tells us that the first condition $\psi_i(F(S)) = F(S)$ of Definition 1.1 holds obviously if a semigroup is generated by (semi) conjugate functions.

Proof of Theorem 2.1. The first part for an almost abelian semigroup follows from Lemma 2.1. The hypothesis $\psi \circ f_i = f_j \circ \psi\ \forall\ i \neq j$ gives $f \circ \psi = \psi \circ g\ \forall\ f,\ g \in S$. From the hypothesis $\psi \circ f_i = f_j\ \forall\ i \neq j$, we can get $\psi \circ f = g, \forall f, g \in S$. Therefore, $\psi \circ g \circ f = f \circ \psi \circ f = f \circ g\ \forall\ f,\ g \in S$. □

Example 2.1. Let $\psi(z) = -z + c$ for some $c \in \mathbb{C}$. Let f be a transcendental entire function such that $f \circ \psi = f$, and define a function g by $g = \psi \circ f$. Then functions f and g are conjugates, and the semigroup $S = \langle f, g \rangle$ is almost abelian.

Solution. Let $f,\ g$, and ψ be as in the statement of the question. It is clear that $\psi^2 =$ identity. Then $g \circ \psi = \psi \circ f \circ \psi = \psi \circ f$. This proves that functions f and g are conjugates. The condition $\psi(F(S)) = F(S)$ for $\psi \in \Psi$ of Definition 1.1 follows from Lemma 2.1. The second condition follows from Theorem 2.1. □

Example 2.2. Let $g(z) = e^{z^2} + \gamma$ and $f = \psi \circ g$, where $\psi(z) = -z$. Then the semigroup $S = \langle f, g \rangle$ is almost abelian.

Solution. The given functions in the question fulfill all conditions such as $f \circ \psi = f$, $\psi^2 =$ identity as well as $\psi \circ g = f \circ \psi$ of Theorem 2.1 and Example 2.1. Therefore, the semigroup $S = \langle f, g \rangle$ is almost abelian. Note that $\psi \circ f = -f \neq f$, so ψ is not an identity. $\square$

Note that Example 2.1 is just for a nice general example of Theorem 2.1, and it says there is an almost abelian transcendental semigroup. Unfortunately, this example does not generate many more examples of transcendental entire functions that can generate transcendental semigroup. Basically, it generates even functions or translates of even functions.

3 Proof of Proposition 1.1

The principal feature of the Julia and Fatou sets of certain abelian transcendental semigroups were studied by Poon[12], where they investigated Julia and Fatou's sets of a complex semigroup coincide with Julia and Fatou's sets of every cyclic subsemigroups.

Definition 3.1. For any analytic map g, the *critical value* and the *asymptotic value* of g are defined by $CV(g) = \{w \in \mathbb{C} : w = g(z) \text{ with } g'(z) = 0\}$ and $AV(g) = \{w \in \mathbb{C} : g(\Gamma(t)) \to w \text{ as } t \to \infty \text{ for any curve } \Gamma = \Gamma(t) \to \infty\}$ repectively. The *singular values* of g is defined by $SV(g) = \overline{(CV(g) \cup AV(g))}$. We say that g is said to be of the finite type or bounded type depending on whether $SV(g)$ is finite or bounded.

Definition 3.2. Let S be a transcendental semigroup. It is said to be finite or bounded type depending on whether each $f \in S$ is of finite or bounded type.

As we mentioned before, in particular, Poon [12, Theorem 5.1] proved the result as shown below.

Theorem 3.1. *For any abelian transcendental semigroup $S = \langle f_1, f_2, \dots f_n \rangle$ of finite type, we have $F(S) = F(f) \,\forall\, f \in S$.*

Indeed, this result looks like an extension work of the following results of Singh and Wang [13, Theorems 2, 3] of classical transcendental dynamics.

Theorem 3.2. *Let g and h be transcendental entire functions without wandering domains such that $g \circ h = h \circ g$. Then $J(g) = J(g \circ h) = J(h)$.*

Theorem 3.3. *Let g and h be transcendental entire functions of the bounded type such that $g \circ h = h \circ g$. Then $J(g) = J(g \circ h) = J(h)$.*

Hinkkanen and Martin [3, Theorem 4.1] showed that the Julia set of the almost abelian rational semigroup coincides with the Julia set of each of its cyclic subsemigroups. Indeed, this is a generalization of the result of the abelian rational semigroup that we proved in [17, Theorem 4.1].

Our particular interest is how far the result of Poon [12, Theorem 5.1] can be generalized to almost abelian transcendental semigroups, and Proposition 1.1 is a generalization in this direction.

Lemma 3.1. *For any complex semigroup S, the following assertions hold*

1. $F(S) \subset F(g) \,\forall\, g \in S$.
2. $J(g) \subset J(S) \,\forall\, g \in S$.
3. $I(S) \subset I(g) \,\forall g \in S$.

Proof. The proof is trivial by the definition. $\square$

Lemma 3.2. *Let S be a transcendental semigroup such that $I(S) \neq \emptyset$. Then*

1. *$I(S)^\circ \subset F(S)$ and $I(S)^e \subset F(S)$, where $I(S)^\circ$ and $I(S)^e$ respectively denote the interior and exterior of $I(S)$.*

2. *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

Proof. We refer [8, Lemma 4.2] for the proof of (1). The proof of (2) is obvious. □

Lemma 3.3. *Let f and g be two entire transcendental functions of finite type such that $\psi \circ g \circ f = f \circ g$, where $\psi(z) = cz + d \; \forall \; c, d \in \mathbb{C}$ and $|c| = 1$. Then $F(f) = F(g)$.*

Proof. By hypothesis, f and g have no Baker and wandering domains ([6, Theorems 4.29 and 4.32]). That is, there are subsequences (f^{n_i}) and (g^{n_i}) which do not diverge to ∞ for all z in their respective Fatou sets. We have to prove that $g(F(f)) \subset F(g)$. The reverse inclusion $F(g) \subset F(f)$ holds similarly.

Consider a neighborhood U of a point $z_0 \in F(f)$ such that $\overline{U} \subseteq F(f)$. Then $g(U)$ is a neighborhood of $g(z_0)$. Consider a sequence (f^n) of iterates of f on $g(U)$. By assumption, there is a subsequence (f^{n_i}) of (f^n) that converges to an analytic function $h : U \to \mathbb{C}$. In such a case, f^{n_i} converges to $g \circ h$ on $g(U)$, that is, $g \circ f^{n_i} \to g \circ h = \xi$ (say) uniformly on U. By assumption, $f^{n_i} \circ g = \psi \circ g \circ f^{n_i} \to \psi \circ \xi = \varphi$ uniformly on U. This proves that $\{f^n \circ g : n \in \mathbb{N}\}$ is a normal family on U, and so $\{f^n : n \in \mathbb{N}\}$ is a normal family on $g(U)$. Since $F(f)$ is a maximal open set where $\{f^n : n \in \mathbb{N}\}$ is normal. Therefore, $g(U) \subseteq F(f)$. This proves that $g(z_0) \in F(f)$, and hence $g(F(f)) \subset F(g)$. □

Lemma 3.4. *Let f and g be finite type entire transcendental functions such that $f \circ g = \psi \circ g \circ f$, where $\psi(z) = cz + d \; \forall$ and $|c| = 1$. Then $I(f) = I(g)$.*

Proof. By Lemma 3.3, $J(f) = J(g)$. By [17, Lemma 4.1], the rest of the proof of this lemma follows. □

Proof of Proposition1.1. We prove $I(S) = I(f) \; \forall \; f \in S$ and the remaining equalities follow from Lemma 3.2. The semigroup S is almost abelian, so, $\forall \; f, g \in S$, there is $\psi \in \Psi$ such that $\psi \circ g \circ f = f \circ g$. Also, $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $g = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$, where each f_{i_k}, f_{j_k} is of finite type. By [12, Lemma 5.1], every f and g in S is of finite type. By Lemma 3.4, we have $I(f) = I(g) \; \forall \; f, g \in S$. By Lemma 3.1(3), $I(S) = \bigcap_{f \in S} I(f)$. Therefore, this fact together with the fact $I(f) = I(g) \; \forall \; f, g \in S$ of Lemma 3.4, we can conclude the assertion of Proposition 1.1. □

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