Characterization of Sets K for Which $H^\infty_K(\mathbb{D})$ is an Algebra-Part II

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Abstract: *Let $K \subset \mathbb{Z}_+$ and define the set $H^\infty_K(\mathbb{D})$ to be the collection of all bounded analytic functions on the unit disk $\mathbb{D}$ in the complex plane whose k^{th} derivative vanishes at zero for all $k \in K$. Depending on the choice of K, the set $H^\infty_K(\mathbb{D})$ may or may not be an algebra. We consider the case where K is infinite and show how to construct these sets.*

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1 Introduction

Let $H^\infty(\mathbb{D})$ be the algebra of all bounded analytic functions that map the unit disk $\mathbb{D}$ into the complex plane $\mathbb{C}$. The set that we are interested in is described as follows. Let $K \subset \mathbb{Z}_+$ and define the set $H^\infty_K(\mathbb{D})$ of all functions in $H^\infty(\mathbb{D})$ whose k^{th} derivative vanishes at zero; formally, we define

$$H^\infty_K(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) \mid f^{(k)}(0) = 0, \text{ for all } k \in K\}.$$

When $K = \emptyset$, we define $H^\infty_\emptyset(\mathbb{D}) := H^\infty(\mathbb{D})$. Certain choices of K will cause $H^\infty_K(\mathbb{D})$ to be a subalgebra of $H^\infty(\mathbb{D})$, but there are sets K where this property fails; for example, $H^\infty_{\{2\}}(\mathbb{D})$ is not an algebra. We have given a characterization for all sets that yield algebras in [2]; namely, that K yields an algebra precisely when $\mathbb{Z}_+ \setminus K$ is an abelian semigroup. We also explored in that paper the case where K is finite: a notable result is that K yields an algebra if and only if $\mathbb{N} \setminus K$ is a numerical semigroup, $\mathbb{N}$ denotes the set of nonnegative integers. Finally, we offered an algorithm on how to construct these finite sets. These results in addition to other facts about K are presented in section 1 below. We will then introduce the Frobenius number and prove properties concerning the generalized Frobenius number. Lastly, attributes of the stabilization point of an infinite algebra yielding set are proven. Sections 2 and 3 offer a procedure for constructing all infinite sets that yield algebras, thereby completing our quest in answering the question posted in Chapter 6.4 of [5].

1.1 Preliminary results

In this section, we will review pertinent results related to our investigation. The following theorem is a characterization of all sets K that yield algebras (see Theorem 3.1 in [2]).

Theorem 1.1 (Banjade and Dunivin)**.** *Let $K \subset \mathbb{Z}_+$. Then $H^\infty_K(\mathbb{D})$ is an algebra if and only if $\mathbb{Z}_+ \setminus K$ is closed under addition.*

As a consequence, the finite case reveals a connection with numerical semigroups, see Corollary 3.1 in [2].

Theorem 1.2 (Banjade and Dunivin)**.** *If $K \subset \mathbb{Z}_+$ is finite, then K yields an algebra precisely when $\mathbb{N} \setminus K$ is a numerical semigroup.*

The following sets were defined in [2] to describe the family $\mathscr{K}$ of all finite sets K that yield algebras.

(a) The set $[k] := \{1, 2, \dots, k\}$ denotes the collection of the first k positive integers.

(b) Let $A \subset \mathbb{Z}$ be a nonempty set. The set

$$S(A)_{\leq k} := \{a + a^\star \mid a, a^\star \in A\}$$

is the collection of all sums of integers in A that are at most k. We note that if $A \subset [k-1]$ is nonempty, then $[k] \setminus A$ yields an algebra precisely when $\mathcal{S}(A)_{\leq k} \subset A$ (see Theorem 4.1 in [2]).

(c) Let $k \geq 3$ and $r \in \big[\, \lfloor (k-1)/2 \rfloor \,\big]$. Define the family

$$\mathcal{J}_r := \left\{ J \in \mathcal{P}\big(\, [k-1] \,\big) \setminus \{\emptyset\} \;\middle|\; \begin{array}{l} \text{i. } |J| = r \\ \text{ii. } \mathcal{S}(J)_{\leq k} \subset J \end{array} \right\}$$

and let

$$\mathscr{K} := \left\{ \emptyset,\, [1],\, [2],\, [k],\, [k] \setminus J \;\middle|\; k \geq 3,\, r \in \left[\, \left\lfloor \frac{k-1}{2} \right\rfloor \,\right],\, J \in \mathcal{J}_r \right\}.$$

We then have the following result (see Theorem 4.2 in [2]).

Proposition 1.1 (Banjade and Dunivin). *The set $\mathscr{K}$ contains every finite set that yields an algebra.*

An algorithm for building the sets in $\mathscr{K}$ arises naturally from the definitions of $\mathcal{J}_r$ and $\mathscr{K}$ (see Algorithm 1 in [2]). If we let $\mathcal{K}^\infty$ be the family of all infinite sets that yield algebras, our goal is to provide a procedure for building its members.

Ryle conducted a brief analysis on the structure of the set K when it yields an algebra. We catalogue some of these results that we have drawn upon in this work (see Lemmas 4.2.1, 4.2.2, 4.2.6, and 4.2.7 in [5]).

Lemma 1.1 (Ryle). *Suppose that K is a nonempty set of positive integers such that $H_K^\infty(\mathbb{D})$ is an algebra.*

(a) The integer $k \notin K$ if and only if $z^k \in H_K^\infty(\mathbb{D})$.

(b) If $j, k \notin K$ where $\gcd(j,k) = 1$, then K is finite.

(c) If $j, k \notin K$ with $\gcd(j,k) = d$, then there exists C_d such that $Nd \notin K$ for any $N \geq C_d$.

(d) Suppose $j, k \notin K$. Then $j + k \notin K$.

The following theorems reveal the form of the objects contained in $H_K^\infty(\mathbb{D})$ (see Lemma 2 and Corollary 1 in [6]).

Theorem 1.3 (Ryle and Trent). *There exists $d \in \mathbb{Z}_+$ such that the following hold.*

(a) If $z^p \in H_K^\infty(\mathbb{D})$, then $p = md$.

(b) There is an $N_0 \in \mathbb{Z}_+$ such that $(z^d)^n \in H_K^\infty(\mathbb{D})$ for all $n \geq N_0$.

Theorem 1.4 (Ryle and Trent). *If $H_K^\infty(\mathbb{D})$ is an algebra, then there exists $d \in \mathbb{Z}_+$, a finite set $n_1 < \cdots < n_p$ in $\mathbb{Z}_+$ with $\gcd(n_1, \ldots, n_p) = 1$, and a positive integer $N_0 > n_p$ so that*

$$\mathbb{Z}_+ \setminus K = \{n_1 d, n_2 d, \ldots, n_p d, N_0 d, (N_0 + j)d : j \in \mathbb{Z}_+\}.$$

We note that Theorem 1.3 was originally introduced as Theorems 4.1 and 4.2 in [5], but it is an improvement since Theorem 1.3 uses fewer assumptions, in particular, it does not require K to be infinite. Further, Theorems 4.1 and 4.2 were established by contradiction while Theorem 1.3 was proven directly using algebraic techniques. However, we found an alternative proof of Theorem 1.3. Due to its brevity and the fact that it contains strategies that will be used in various proofs throughout our work, we have chosen to include it.

Proof of Theorem 1.3. Choose $a, b \in \mathbb{Z}_+ \setminus K$ such that $d := \gcd(a,b)$ is the smallest greatest common divisor. By Lemma 1.1.(c), there is a positive integer N such that $nd \in \mathbb{Z}_+ \setminus K$ for all $n \geq N$. Let $k \in \mathbb{Z}_+ \setminus K$. Pick a prime $p > \max\{N, k\}$ and set $d_1 := \gcd(k, pd)$. Then $pd \in \mathbb{Z}_+ \setminus K$. As $k, pd \in \mathbb{Z}_+ \setminus K$, we have $d \leq d_1$. But $d_1 \leq k < p$, and thus $\gcd(d_1, p) = 1$. Consequently, d_1 divides d, so $d_1 = d$. Hence, d divides k. ■

1.2 Generalized Frobenius number

The Frobenius coin problem is stated as follows.
Given positive integers $a_1, a_2, \ldots, a_n$ such that $\gcd(a_1, a_2, \ldots, a_n) = 1$, find the largest integer that cannot be expressed as a linear combination $k_1a_1 + k_2a_2 + \cdots + k_na_n$, where $k_1, k_2, \ldots, k_n$ are nonnegative integers (see [1] and [4] for a history of this famous problem). Restricting ourselves to the two variable case, the following Theorem gives a formula for the largest positive integer that cannot be written as a linear combination of two co-prime integers (see [1]). We note that J. J. Sylvester was the first to give the following formula as a solution to the two variable coin problem (see [1] and [7]). An immediate consequence of the theorem is an exact count of all positive integers that cannot be written as a linear combination of two given integers over the nonnegative integers.

Theorem 1.5 (Frobenius Theorem)**.** *For any two relatively prime positive integers a and b, the greatest integer that cannot be written in the form $am + bn$ for nonnegative integers m and n is $ab - a - b$.*

Corollary 1.1. *Suppose that a and b are relatively prime. Then there are exactly $\dfrac{(a-1)(b-1)}{2}$ positive integers that cannot be written as $am + bn$ for nonnegative integers m and n, not both zero.*

The number $ab-a-b$ is known as the Frobenius number. However, the number that captures our attention is the following generalized version of the Frobenius number.

Definition 1.1. Let $a, b \in \mathbb{Z}_+$. Define the **Generalized Frobenius Number** $\ell(a, b)$ by

$$\ell(a, b) := \operatorname{lcm}(a, b) - a - b.$$

The next proposition extends the Frobenius Theorem and serves as a keystone for much of our work.

Proposition 1.2. *The number $\ell(a, b)$ is the largest multiple of $\gcd(a, b)$ that cannot be written as $am + bn$ for some nonnegative integers m and n, not both zero.*

Proof. See [1] for a proof. ∎

Proposition 1.3. *Let $a, b \in \mathbb{Z}_+$ such that $a \le b$. Let $d = \gcd(a, b)$. The number of multiples of d that are greater than a and cannot be written as $am + bn$ where m and n are nonnegative integers not both zero is* $\dfrac{(a/d-1)(b/d-1)}{2} - \dfrac{a}{d} + 1$.

Proof. We know that a/d and b/d are relatively prime, so Corollary 1.1 tells us that there are $\frac{(a/d-1)(b/d-1)}{2}$ positive integers that cannot be written as $(a/d)m + (b/d)n$ where m and n are nonnegative integers. Since no positive integers less than a/d can be written as such a linear combination, we can remove them from our count, so that the number of integers at least as large as a/d that cannot be written as our desired linear combination is $\frac{(a/d-1)(b/d-1)}{2} - a/d + 1$. As there is a bijection between an integer and its product with d, the conclusion follows. ∎

Although the above attributes of the generalized Frobenius number $\ell(a, b)$ will be useful in our work, we still need to know more about it. Since we were unable to find literature providing the fundamental properties of $\ell(a, b)$, we found it interesting in and of itself to discover more of its characteristics. In particular, the following proposition captures the relationship between $\ell(a, b)$ and the integers a and b. Note that $\ell(a, b) \neq a, b$ by Proposition 1.2.

Proposition 1.4. *Let $a, b \in \mathbb{Z}_+$ such that $a < b$. Let $d = \gcd(a, b)$.*

1. *The following statements are equivalent:*
   (i) $\ell(a, b) < 0$.
   (ii) $a = d$.
   (iii) $\ell(a, b) = -a$.

2. *The number $\ell(a,b)$ is nonzero.*
3. *The following are equivalent:*
   (i) $0 < \ell(a,b) < a$.
   (ii) $a = 2d$ *and* $b = 3d$.
   (iii) $\ell(a,b) = d$.
4. *We have* $a < \ell(a,b) < b$ *if and only if* $a = 2d$ *and* $b = a + md$ *for some odd* $m > 1$.
5. *We have* $b < \ell(a,b)$ *if and only if* $a = qd$ *and* $b = a + md$ *for some* $q \geq 3$ *and* $m \in \mathbb{Z}_+ \setminus q\mathbb{Z}_+$. *Moreover,* m *and* q *are relatively prime.*
6. *Let* $c, m, n \in \mathbb{Z}_+$ *such that* c *divides* m. *Then*
$$\ell(mn, (m+c)n) = \frac{n}{c}\Big(m - c\phi\Big)\Big(m + \frac{c}{\phi}\Big),$$
where $\phi = \dfrac{1+\sqrt{5}}{2}$ *is the golden ratio and* $\dfrac{1}{\phi} = \dfrac{\sqrt{5}-1}{2}$ *is the silver ratio.*

Proof.

1. Assume $\ell(a,b) < 0$. Then $ab/d < a + b < 2b$, and thus $a/d < 2$. Hence, $a = d$. Now, if $a = d$, it is obvious that $\ell(a,b) = -a$. Finally, if $\ell(a,b) = -a$, then $\ell(a,b) < 0$.
2. If $\ell(a,b) = 0$, then $ab/d = a + b < 2b$, implying that $\frac{a}{d} = 1$ as $a, d > 0$. This gives us $\ell(a,b) = a + 2b$, contradicting Proposition 1.2.
3. Suppose that $0 < \ell(a,b) < a$. Then $ab/d < 2a + b < 3b$, so $a/d = 1, 2$. By part 1, it follows that $a = 2d$. Let $b = a + md$ for some positive integer m. Then $\ell(a,b) = md < 2d = a$, so $m = 1$, and thus $b = 3d$. If $a = 2d$ and $b = 3d$, it is obvious that $\ell(a,b) = d$. Finally, if $\ell(a,b) = d$, then $\ell(a,b) \leq a$ since d divides a. Further, $\ell(a,b) > 0$, so by part 1, $a \neq d$. Hence, $0 < \ell(a,b) < a$.
4. Suppose $a < \ell(a,b) < b$. A similar argument as in part 3 yields $a = 2d$. Write $b = a + md$ for some $m \in \mathbb{Z}_+$. Then $\ell(a,b) = md$. As $\ell(a,b)$ does not divide a, we have $m > 1$ and odd.
5. Suppose $b < \ell(a,b)$ Write $a = qd$ and $b = a + md = (q+m)d$ for some integers m and q. A direct calculation shows that $\ell(a,b) = (q-1)(q+m)d - qd > (q+m)d$. The equation of $\ell(a,b)$ shows that $m \notin q\mathbb{Z}_+$ since $\ell(a,b)$ does not divide a. The inequality shows that $q \neq 1, 2$.
6. Since c divides m, we have that $\gcd(mn, (m+c)n) = cn$. Thus,
$$\begin{aligned}
\ell(mn, (m+c)n) &= \frac{mn(m+c)n}{cn} - mn - (m+c)n \\
&= \frac{n}{c}\big(m^2 - cm - c^2\big) \\
&= \frac{n}{c}\Big(m^2 - cm + \frac{c^2}{4} - \frac{c^2}{4} - c^2\Big) \\
&= \frac{n}{c}\Big[\Big(m - \frac{c}{2}\Big)^2 - \frac{5c^2}{4}\Big] \\
&= \frac{n}{c}\Big(m - c\cdot\frac{1+\sqrt{5}}{2}\Big)\Big(m + c\cdot\frac{\sqrt{5}-1}{2}\Big) \\
&= \frac{n}{c}\Big(m - c\phi\Big)\Big(m + \frac{c}{\phi}\Big).
\end{aligned}$$

∎

1.3 Stabilization point

Ryle also introduced the notion of an infinite set K that yields an algebra stabilizing at some point Nd (see Definition 4.2.1 in [5]). This definition crystallizes the idea that every member in $\mathbb{Z}_+ \setminus K$ will eventually be consecutive integer multiples of d due to Theorems 1.3 and 1.4. In light of Lemma 1.1.(a), we present an adapted yet equivalent version of her definition.

Definition 1.2. For $N, d \in \mathbb{Z}_+$, we say that the infinite set $K \subset \mathbb{Z}_+$ **stabilizes** at Nd if

(a) $(N-1)d \in K$ where $N > 1$,

(b) $nd \in \mathbb{Z}_+ \setminus K$ for all $n \geq N$, and

(c) if $v \in \mathbb{Z}_+ \setminus K$ where $v \geq Nd$, then d divides v.

We call Nd a **stabilization point** of K. If $N = 1$ and conditions (b) and (c) are satisfied, then K **stabilizes immediately** at d.

The following example illustrates the idea of stabilization and was originally presented in [5].

Example 1.1. Consider the infinite sets

$$K = \{1, \dots, 5, 7, 9, 10, 11, 2n+1 \,|\, n \geq 6\}$$
$$\mathbb{Z}_+ \setminus K = \{6, 8, 2n \,|\, n \geq 6\}$$

Clearly, K yields an algebra by Theorem 1.1. We see that $d = 2$ and $N = 6$, so that $(N-1)d = 10 \in K$, fulfilling condition (a) in the definition. Moreover, $nd = n \cdot 2 \in \mathbb{Z}_+ \setminus K$ for all $n \geq N$, so condition (b) is fulfilled. Lastly, we know from Theorems 1.3 and 1.4 that condition (c) holds. Therefore, $Nd = 6 \cdot 2 = 12$ is a stabilization point of K.

In light of Theorems 1.3 and 1.4, it is obvious that every infinite set $K \subsetneq \mathbb{Z}_+$ that yields an algebra must stabilize. Nevertheless, we shall give a proof of this fact, but we will also show that K stabilizes at only one point, and that the factors used to construct the stabilization point are unique.

Proposition 1.5. *Suppose that $K \subsetneq \mathbb{Z}_+$ is infinite and yields an algebra. Then there exist unique integers N and d such that K stabilizes at Nd.*

Proof. Since K is infinite and yields an algebra, Theorem 1.4 says that there are positive integers d and $n_1 < n_2 < \cdots < n_j < N$ such that

$$\mathbb{Z}_+ \setminus K = \{n_1 d, n_2 d, \dots, n_j d, Nd, (N+i)d \,|\, i \in \mathbb{Z}_+\}.$$

We can assume that N is the smallest integer such that $Nd, (N+i)d \in \mathbb{Z}_+ \setminus K$ for all i. Now, if $N = 1$, then $\mathbb{Z}_+ \setminus K = d\,\mathbb{Z}_+$, so K stabilizes at $Nd = d$. Suppose that $N > 1$. Then n_j and N are nonconsecutive due to the minimality of N, so $(N-1)d \notin \mathbb{Z}_+ \setminus K$; and since $N - 1 > 0$, we deduce that $(N-1)d \in K$. Therefore, K stabilizes at Nd.

Now suppose that K also stabilizes at Me. We first show that $d = e$. Select a prime $r > \max\{M, Nd\}$. Because Me is a stabilization point and $r > M$, we have $re \in \mathbb{Z}_+ \setminus K$. But Nd is also a stabilization point, so d divides re. Since $r > d$ is prime, we have $\gcd(r, d) = 1$, and thus d divides e. Similarly, e divides d, so we conclude that $e = d$. Now, the minimality of N implies that $N \leq M$. However, if $N < M$, then $(M-1)e = (M-1)d \in K$ by Definition 1.2.(a). But the fact that $N \leq M - 1$ implies that $(M-1)d \notin K$ by Definition 1.2.(c), a contradiction. Hence, $M = N$, completing the proof. ■

We know from Theorem 1.3 that integers N and d exist possessing certain properties. We improve upon these results by giving an explicit description of the integer d, as well as providing sharp upper and lower bounds for N when K does not stabilize immediately (if K stabilizes immediately, then $N = 1$, by definition). We begin by dealing with d. Consider the following example.

Example 1.2. The following sets yield algebras, as the reader can confirm using Theorem 1.1.

(a) Let $a = 8$ and $b = 10$. Consider the infinite set K satisfying

$$\mathbb{Z}_+ \setminus K = \{8, 10, 16, 18, 20, 2n \,:\, n \geq 12\}.$$

We see that the smallest greatest common divisor $d = \gcd(8, 10) = 2$. Also note that $8 = \min(\mathbb{Z}_+ \setminus K)$ and $10 = pd$ where $p = 5$ is prime.

(b) Let $a = 6$ and $b = 21$. Suppose K is given such that $\mathbb{Z}_+ \setminus K = \{3n \,:\, n \geq 2\}$. The smallest greatest common divisor $d = \gcd(6, 9) = 3$. Further, 6 is the minimum of $\mathbb{Z}_+ \setminus K$, and $9 = pd$ where $p = 3$ is prime.

The above examples suggest that if K stabilizes at Nd, then d is the smallest greatest common divisor of some pair of integers in $\mathbb{Z}_+ \setminus K$. We now prove that this is not a coincidence.

Proposition 1.6. *Let $K \subsetneq \mathbb{Z}_+$ be an infinite set that yields an algebra and stabilizes at Nd. Let $m := \min(\mathbb{Z}_+ \setminus K)$. Then there is a $c \in \mathbb{Z}_+ \setminus K$ such that $d = \gcd(m, c) \leq \gcd(i, j)$ for all $i, j \in \mathbb{Z}_+ \setminus K$. Moreover, if $m > d$, then m does not divide c.*

Proof. Select a prime $p > \max\{m, N\}$. Since K stabilizes at Nd and $p > N$, we have $pd \in \mathbb{Z}_+ \setminus K$. We claim that pd is our desired integer. Let $m = qd$ for some $q \in \mathbb{Z}_+$. Since $p > q$ is prime, $\gcd(p, q) = 1$. Therefore, $\gcd(m, pd) = d$. Now let d_1 denote the smallest gcd of some $i, j \in \mathbb{Z}_+ \setminus K$. Then $d_1 \leq d$. However, since K stabilizes at Nd and $i, j \in \mathbb{Z}_+ \setminus K$, we have d divides i and j, and thus d divides $\gcd(i, j) = d_1$. Hence, $d = d_1$. Finally, suppose that $m > d$. As $p > m$ is prime and d divides m, it follows that m divides pd if and only if $m = d$. Therefore, m does not divide pd. ∎

We are now interested in providing sharp upper and lower bounds for N when K does not stabilize immediately. The next two propositions deal with this situation, but we first consider some examples concerning an inequality on N.

Example 1.3. We begin with $\mathbb{Z}_+ \setminus K = \{12, 24, 28, 36, 40, 4n \,|\, n \geq 12\}$, which stabilizes at $4N = 4 \cdot 12$. Note that $m = \min(\mathbb{Z}_+ \setminus K) = 12$ and that the smallest greatest common divisor $d = 4$. The smallest member $m^\star \in \mathbb{Z}_+ \setminus K$ such that $d = \gcd(m, m^\star)$ is $m^\star = 28$. Thus, $\ell(m, m^\star) = 44$. We now make the following observations involving N and $\ell(m, m^\star)$:

(a) $\ell(m, m^\star) \notin \mathbb{Z}_+ \setminus K$.

(b) $N = 12 = \ell(m, m^\star)/d + 1$.

(c) $m/d \leq N \leq \ell(m, m^\star)/d + 1$.

In this case, N achieves its maximum value in the inequality of (c).

Example 1.4. Now let us form $(\mathbb{Z}_+ \setminus K) \cup \{\ell(m, m^\star)\}$. We then obtain the set $\mathbb{Z}_+ \setminus K = \{12, 24, 28, 4n \,|\, n \geq 9\}$, which stabilizes at $4N = 4 \cdot 9$. We thus make the following observations (the values of m, $m^\star$, d and $\ell(m, m^\star)$ are the same):

(a) $\ell(m, m^\star) \in \mathbb{Z}_+ \setminus K$.

(b) $m/d \leq N = 9 \leq 12 = \ell(m, m^\star)/d + 1$.

So N does not achieve its maximum value, but it still satisfies the inequality in (b).

Example 1.5. Finally, the integers 16, 20, 32, and $44 = \ell(m, m^\star)$ are all multiples of d greater than $m = 12$ that *cannot* be expressed as $mx + m^\star y$, where $x, y \in \mathbb{Z}_+ \cup \{0\}$. Form the set $(\mathbb{Z}_+ \setminus K) \cup \{16, 20, 32, 44\}$. Then we have $\mathbb{Z}_+ \setminus K = \{4n \,|\, n \geq 3\}$, which stabilizes at $4N = 4 \cdot 3$. Although $m^\star = 16$ now, the following observations are still consistent with what we have above:

(a) All multiples of $d = 4$ greater than or equal to $m = 12$ are members of $\mathbb{Z}_+ \setminus K$.

(b) $N = 3 = m/d$.

(c) $m/d \leq N \leq 6 = \ell(m, m^\star)/d + 1$.

In this case, N achieves its minimum value of m/d when all the multiples of d greater than m that cannot be written as a linear combination of m and $m^\star$ are members of $\mathbb{Z}_+ \setminus K$.

The above examples motivate the next two propositions. We first observe that if K does not stabilize immediately, Proposition 1.6 permits us to deduce that a smallest integer $m^\star \in \mathbb{Z}_+ \setminus K$ exists such that $d = \gcd(m, m^\star)$, $m < m^\star$, and m does not divide $m^\star$, where $m := \min(\mathbb{Z}_+ \setminus K)$ and d is the smallest greatest common divisor.

Proposition 1.7. *Let $K \subsetneq \mathbb{Z}_+ \setminus K$ be infinite and yield an algebra. Suppose that K stabilizes at $Nd = N\gcd(m, m^\star)$. Then $m/d \leq N \leq \ell(m, m^\star)/d + 1$.*

Proof. Since K stabilizes at Nd, we know $Nd \in \mathbb{Z}_+ \setminus K$. As $m := \min(\mathbb{Z}_+ \setminus K)$, we have $m \leq Nd$. Now, assume to the contrary that $\ell(m, m^\star)/d + 1 < N$, so that $\ell(m, m^\star) + d < Nd$. Then $(N-1)d = Nd - d > \ell(m, m^\star)$. By Proposition 1.4, there exist nonnegative integers x and y not both zero such that $(N-1)d = mx + m^\star y$. Since $m, m^\star \in \mathbb{Z}_+ \setminus K$ and $\mathbb{Z}_+ \setminus K$ is closed under addition by Theorem 1.1, $(N-1)d \in \mathbb{Z}_+ \setminus K$. However, as $N - 1 \geq 1$ and Nd is a stabilization point of K, we have $(N-1)d \in K$, a contradiction. Therefore, $N \leq \ell(m, m^\star)/d + 1$. ■

The following proposition explains when N achieves its maximum and minimum values.

Proposition 1.8. *Let $K \subsetneq \mathbb{Z}_+ \setminus K$ be infinite and yield an algebra. Suppose that K stabilizes at Nd with $N > 1$.*

(a) If $\ell(m, m^\star) \notin \mathbb{Z}_+ \setminus K$, then $N = \ell(m, m^\star)/d + 1$.

(b) If $\mathbb{Z}_+ \setminus K$ contains all multiples of d greater than m that cannot be written as $mx + m^\star y$ for some nonnegative integers x and y not both zero, then $N = m/d$.

Proof.

(a) Since $\big(\ell(m, m^\star)/d + 1 - 1\big)d = \ell(m, m^\star) \in K$, it follows that $\ell(m, m^\star)/d + 1$ satisfies Definition 1.2.(a). Further, Proposition 1.2 tells us that $nd \in \mathbb{Z}_+ \setminus K$ for all $n \geq \ell(m, m^\star)/d + 1$, so $\ell(m, m^\star)/d + 1$ fulfills Definition 1.2.(b). Finally, suppose $v \in \mathbb{Z}_+ \setminus K$ such that $v \geq \ell(m, m^\star) + d$. Then d divides v since every member of K is a multiple of d. Therefore, $(\ell(a, b)/d + 1)d$ is a stabilization point of K, so $N = \ell(m, m^\star)/d + 1$ by Proposition 1.5.

(b) First, we see that $(m/d - 1)d \in K$ since $m = \min(\mathbb{Z}_+ \setminus K)$, so Definition 1.2.(a) is satisfied. From our assumption we deduce that every multiple of d greater than or equal to $m = (m/d)d$ is a member of $\mathbb{Z}_+ \setminus K$, so Definition 1.2.(b) is fulfilled. The third property of Definition 1.2 is automatically accomplished because K contains only multiples of d. By Proposition 1.5, we conclude that $N = m/d$.

■

2 Construction of Infinite Sets That Yield Algebras

Similar to the family $\mathscr{J}_r$ that was defined in [2] in order to construct the set $\mathscr{K}$ of all finite sets that yield algebras, we now define a family of sets that will be utilized in building the family of all infinite sets that yield algebras.

Definition 2.1. For each positive integer $d > 1$, define the family $\mathscr{J}_d$ by

$$\mathscr{J}_d = \left\{ J \in \mathcal{P}(\mathbb{Z}_+) \;\middle|\; \begin{array}{l} \text{i. there exist } m, m^\star \in J \text{ such that } d = \gcd(m, m^\star) \\ \text{ii. for all } j_1, j_2 \in J,\ d \leq \gcd(j_1, j_2) \\ \text{iii. } J \text{ is closed under addition} \end{array} \right\}$$

Note that the family $\mathcal{J}_d \neq \emptyset$ for all $d > 1$ since the set $d\,\mathbb{Z}_+ \in \mathcal{J}_d$. The motivation for the above definition of $\mathcal{J}_d$ is as follows. First, we want to construct infinite sets $K := \mathbb{Z}_+ \setminus J$ that yield algebras; but if there exist $a, b \in J$ such that $\gcd(a, b) = 1$, then K must be finite by Lemma 1.1.(b). Consequently, requiring $d > 1$ and enforcing condition i. are necessary. Further, condition ii. is required because we know from Proposition 1.6 that the smallest greatest common divisor is involved in generating the members of $\mathbb{Z}_+ \setminus K$. Lastly, requiring J to be closed under addition is a necessary and sufficient condition for K to yield an algebra by Theorem 1.1. So our strategy in constructing infinite proper subsets that yield algebras is to first obtain the sets J, and then form the set $K := \mathbb{Z}_+ \setminus J$. We shall see in Proposition 2.2 that all sets in $\mathcal{J}_d$ can be used to build a set that yields an algebra, and that all infinite sets that yield algebras can be written in terms of some member in $\mathcal{J}_d$ for a $d > 1$.

We now consider some examples concerning $\mathcal{J}_d$.

Example 2.1.

(a) Consider the set $3\mathbb{Z}_+ \setminus \{3, 6\}$. We first note that $m = 9$ and $m^\star = 12$, so that $d = \gcd(m, m^\star) = 3$ and $d \le \gcd(j_1, j_2)$ for all $j_1, j_2 \in 3\mathbb{Z}_+ \setminus \{3, 6\}$. Further, $3\mathbb{Z}_+ \setminus \{3, 6\}$ is closed under addition. Hence, $3\mathbb{Z}_+ \setminus \{3, 6\} \in \mathcal{J}_3$. However, the set $3\mathbb{Z}_+ \setminus \{15\} \notin \mathcal{J}_3$ since $6 + 9 \notin 3\mathbb{Z}_+ \setminus \{15\}$.

(b) The set $4\mathbb{Z}_+ \cup 6\mathbb{Z}_+ \notin \mathcal{J}_2$ since $4+6 \notin 4\mathbb{Z}_+ \cup 6\mathbb{Z}_+$. In contrast, the $4\mathbb{Z}_+ + 6\mathbb{Z}_+ = \{4m+6n \,|\, m, n \in \mathbb{Z}_+\}$ is a member of $\mathcal{J}_2$: We see that $m = 10$, $m^\star = 14$, $d = \gcd(a, b) = 2$, $d \le \gcd(j_1, j_2)$ for all $j_1, j_2 \in 4\mathbb{Z}_+ + 6\mathbb{Z}_+$, and is clearly closed under addition.

We now begin proving several properties of the sets $J \in \mathcal{J}_d$ that are similar to those that Ryle established under the assumption that K yields in algebra. Proposition 2.1 below relates to Theorems 1.3 and 1.4.

Proposition 2.1. *Let $d > 1$ and $J \in \mathcal{J}_d$ such that $d = \gcd(a, b)$ for some $a, b \in J$. Then $nd \in J$ for all $n \ge \ell(a, b)/d + 1$. Further, d divides every member of J.*

Proof. Let $a, b \in J$ such that $d = \gcd(a, b)$. If $n \ge \ell(a, b)/d + 1$, then $nd > \ell(a, b)$. By Proposition 1.2, $nd = ax + by$ for some nonnegative integers x and y not both zero. Because J is closed under addition, $nd = ax + by \in J$, as desired. Now let $j \in J$. Choose a prime $p > \max\{j,\, \ell(a, b)/d+1\}$. Then $pd \in J$. Now set $d_1 = \gcd(j, pd)$. Because $j, pd \in J$, it follows from the definition of J that $d \le d_1$. However, $d_1 \le j < p$, and since p is prime, $\gcd(d_1, p) = 1$. Consequently, d_1 divides d, and thus $d_1 = d$. Hence, d divides j. ■

We now show that every collection in $\mathcal{J}_d$ can be used to obtain a set that yields an algebra.

Proposition 2.2. *For all $J \in \mathcal{J}_d$ where $d > 1$, the set $\mathbb{Z}_+ \setminus J$ is an infinite proper subset of $\mathbb{Z}_+$ that yields an algebra.*

Proof. To see that $\mathbb{Z}_+ \setminus J$ is infinite, let $d = \gcd(a, b)$ for some $a, b \in J$. By Proposition 2.1, we see that for each $n \ge \ell(a, b)/d + 1$, the integer $nd + 1 \in \mathbb{Z}_+ \setminus J$ since d does not divide $nd + 1$. Hence, $\mathbb{Z}_+ \setminus J$ is infinite. Moreover, the fact that $d > 1$ is the smallest greatest common divisor of some pair of integers in J implies that there are consecutive integers x and $x + 1$ not in J, so $\mathbb{Z}_+ \setminus J$ is a proper subset of $\mathbb{Z}_+$. Lastly, the complement J of $\mathbb{Z}_+ \setminus J$ is closed under addition by definition. Thus, $\mathbb{Z}_+ \setminus J$ yields an algebra by Theorem 1.1. ■

Define the set $\mathscr{K}^\infty$ by

$$\mathscr{K}^\infty := \left\{ \mathbb{Z}_+,\, \mathbb{Z}_+ \setminus J \,\middle|\, \text{ for all } J \in \bigcup_{d=2}^{\infty} \mathcal{J}_d \right\}.$$

We will now demonstrate our main result in the next theorem. We first state the following result (see Corollary 2.1 in [2]).

Lemma 2.1. *Suppose $H^\infty_K(\mathbb{D})$ is an algebra. If K only contains consecutive integers, then $K = [k]$ for some integer k, or $K = \mathbb{Z}_+$.*

Theorem 2.1. *The set $\mathscr{K}^\infty$ contains every infinite set that yields an algebra.*

Proof. Proposition 2.2 tells us that every set in $\mathcal{J}_d$ yields an algebra. Now let K be an infinite set that yields an algebra. We wish to show that $K \in \mathcal{K}^\infty$, so we will proceed with cases on K. First, suppose that K only contains consecutive integers. Then by Lemma 2.1, $K = \mathbb{Z}_+$, so $K \in \mathscr{K}^\infty$. Now let us assume that K contains at least one pair of nonconsecutive integers. By Theorems 1.3 and 1.4, there exist positive integers d, N, and $n_1 < n_2 < \cdots < n_j$ such that

$$\mathbb{Z}_+ \setminus K = \{\, n_1 d, n_2 d, \dots, n_j d, Nd, (N+i)d \,|\, i \in \mathbb{Z}_+ \,\}.$$

By Proposition 1.6, the integer d is the smallest greatest common divisor of some pair of integers in $\mathbb{Z}_+ \setminus K$. Thus, there are integers $m, m^\star \in \mathbb{Z}_+ \setminus K$ such that $d = \gcd(m, m^\star) \leq \gcd(j_1, j_2)$ for all $j_1, j_2 \in \mathbb{Z}_+ \setminus K$. Furthermore, the fact that K yields an algebra implies that its complement $\mathbb{Z}_+ \setminus K$ is closed under addition by Theorem 1.1. Given all of these facts concerning $\mathbb{Z}_+ \setminus K$, we deduce that $\mathbb{Z}_+ \setminus K \in \mathcal{J}_d$, and thus $K \in \mathscr{K}^\infty$. ∎

Corollary 2.1. *The set $\mathscr{K} \cup \mathscr{K}^\infty$ contains every set that yields an algebra.*

Proof. This claim follows from Theorems 1.1 and 2.1. ∎

3 Algorithm for Generating All Infinite Sets That Yield Algebras

The trivial infinite set that yields an algebra is $\mathbb{Z}_+$, so we will ignore this case in our algorithm. For the nontrivial case, let a and b be distinct positive integers. Define the set $\mathcal{U}$ to be the collection of all positive integer multiples of $\gcd(a, b)$ that cannot be represented as a linear combination of a and b over the nonnegative integers. We will call the members of $\mathcal{U}$ unrepresentables. This set is empty when $\ell(a, b) < 0$; otherwise, its cardinality is given in Proposition 1.3, and its maximum is $\ell(a, b)$. Our algorithm is as follows.

Algorithm: Generating Infinite Sets That Yield Algebras

1. Specify positive integers $a < b$.
2. Let $S_0 := \{ax + by \mid x, y \in \mathbb{N} \text{ with } x, y \text{ not both zero}\}$. Build the set $\mathbb{Z}_+ \setminus S_0$. If $\ell(a, b) < 0$, the algorithm ends. Otherwise, proceed to step 3.
3. For each $U \in \mathcal{P}(\mathcal{U}) \setminus \{\emptyset\}$, let $u_1, \dots, u_n$ denote all elements in U. Define the set S_U to be the collection of all linear combinations of the form $ax_1 + bx_2 + u_1 x_3 + \cdots + u_n x_{n+2}$ where the x_i's are nonnegative integers not all zero. Build the sets $\mathbb{Z}_+ \setminus S_U$.

It is clear that the sets constructed at the end of the above algorithm yield algebras because their complements are closed under addition by construction. Although it is impossible to write a procedure that generates all infinite sets that yield algebras, it is the case that each of these sets is obtainable at the end of some iteration of the algorithm, which we state in the following proposition whose proof is trivial.

Proposition 3.1. *Every nontrivial infinite set that yields an algebra can be built using Algorithm 2.*

Proof. Let $K \subsetneq \mathbb{Z}_+$ be an infinite set that yields an algebra and set $a := \min(\mathbb{Z}_+ \setminus K)$. Proposition 1.6 says that an integer $b \in \mathbb{Z}_+ \setminus K$ exists such that K stabilizes at $Nd = N \gcd(a, b)$. Then $K = \mathbb{Z}_+ \setminus S_0$ if there are no unrepresentables of a and b in $\mathbb{Z}_+ \setminus K$. Otherwise, $K = \mathbb{Z}_+ \setminus S_U$ where U is the set of all unrepresentables of a and b in $\mathbb{Z}_+ \setminus K$. ∎

To conclude this paper, we give some formulas of infinite algebra yielding sets by imposing a condition on the generalized Frobenius number.

Proposition 3.2. *Let $K \subsetneq \mathbb{Z}_+$ be an infinite set that yields an algebra. Let $m := \min(\mathbb{Z}_+ \setminus K)$ and $m^\star > m$ be the smallest member of $\mathbb{Z}_+ \setminus K$ such that $d := \gcd(m, m^\star)$ is minimal. Then $\ell(m, m^\star) < m^\star$ if and only if K has one of the following forms:*

(a) $\mathbb{Z}_+ \setminus d\mathbb{Z}_+$.

(b) $\mathbb{Z}_+ \setminus (d\mathbb{Z}_+ \setminus \{d\})$.

(c) $\mathbb{Z}_+ \setminus \{2d, 4d, \ldots, (q-1)d, nd \mid n \geq q+1\}$ *for some odd* $q > 1$.

(d) $\mathbb{Z}_+ \setminus \{2d, 4d, \ldots, (r-3)d, nd \mid n \geq r\}\}$ *for each* $r = 3, 5, \ldots, q-2, q$ *and for some odd* $q > 1$.

Proof. Suppose that $\ell(m, m^\star) < m^\star$. If $\ell(m, m^\star) < 0$, then Proposition 1.4 says that $m = d$, so $K = \mathbb{Z}_+ \setminus d\mathbb{Z}_+$. Similarly, if $0 < \ell(m, m^\star) < m$, then $m = 2d$ and $m^\star = 3d$, so $K = \mathbb{Z}_+ \setminus (d\mathbb{Z}_+ \setminus \{d\})$. Finally, suppose $m < \ell(m, m^\star) < m^\star$, so that $\ell(a, b) = qd$. We observe that the unrepresentables of m and $m^\star$ are $3d, 5d, \ldots, (q-2)d, qd = \ell(m, m^\star)$. If there are no unrepresentables in $\mathbb{Z}_+ \setminus K$, then $\mathbb{Z}_+ \setminus K$ contains all even multiples of d smaller than $\ell(m, m^\star)$, so $\mathbb{Z}_+ \setminus K = \{2d, 4d, \ldots, (q-3)d, (q-1)d, nd \mid n \geq q+1\}$. If $\mathbb{Z}_+ \setminus K$ contains unrepresentables, let rd denote the smallest one. Then $ax + rd$ for each $x = 1, 2, \ldots, (q-r)/d$ are all unrepresentables greater than rd and are contained in $\mathbb{Z}_+ \setminus K$. Therefore, $\mathbb{Z}_+ \setminus K = \{2d, 4d, \ldots, (r-3)d, nd \mid n \geq r-1\}$. ∎

When $b < \ell(a, b)$, it is much more difficult to list infinite sets that yield algebras. If a general formula for the stabilization point is known, then one would be able to list all infinite sets that yield algebras. However, the stabilization point is dependent on the Frobenius number, which does not have a general formula.

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