Kummer's Theorems, Popular Solutions and Connecting Formulas on Hypergeometric Function

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Abstract: *The hypergeometric series is an extension of the geometric series. The confluent hypergeometric function is the solution of the hypergeometric differential equation* $[\theta(\theta+b-1)-z(\theta+a)]w=0$. *Kummer's first formula and Kummer's second formula are of significant importance in solving the hypergeometric differential equations. Kummer has developed six solutions for the differential equation and twenty connecting formulas during the period of 1865-1866. Each connecting formula consist of a solution expressed as the combination of two other solutions. Recently in 2021, these solutions were extensively used by Schweizer* [13] *in practical problems specially in Physics. Here we extend the connecting formulas obtained by Kummer to obtain the other six solutions* $w_1(z), w_2(z), w_3(z), w_4(z), w_5(z)$ *and* $w_6(z)$ *as the combination of three solutions.*

Keywords: Hypergeometric series, Confluent hypergeometric function, Kummer's formula, Connecting formula

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1 Introduction

So far, numerous works are done in the field of hypergeometric functions. Before proceeding with the main work, we shall state some definitions, Kummer's Theorems and connection formulas in hypergeometric functions.

1.1 Hypergeometric series [11]

The series

$$1+a+a^2+a^3+a^4+\cdots \tag{1}$$

is the geometric series with the common ratio a. In 1655, John Wallis extended the ordinary geometric series to the hypergeometric series in his book Arithematica Infinitorium [11] as

$$1+a+a(a+b)+a(a+b)(a+2b)+a(a+b)(a+2b)(a+3b)+\cdots$$

with the n^{th} term

$$a(a+b)(a+2b)(a+3b)...[a+(n-1)b] \tag{2}$$

When b is replaced by 1 then the n^{th} term in the expression (2) is now called the Pochhammer symbol

$$a_n=a(a+1)(a+2)(a+3)...[a+(n-1)]=\frac{\Gamma(a+k)}{\Gamma(a)} \tag{3}$$

for $k\in Z^+$.
Similarly, Euler [11] introduced the power series expansion of the form

$$1 + \frac{ab}{c}.\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}.\frac{z^3}{3!} + \cdots \tag{4}$$

The expression (4) is denoted by ${}_2F_1(a, b; c; z)$, where a, b and c are rational parameters, is called the hypergeometric function and may undergo several transformations [9]. It is convergent for $|z| < 1$ and for $|z| > 1$, it is divergent. For $Re(c - a - b) > 0, Re(c) > Re(b) > 0$, the hypergeometric function can be represented in terms of gamma function as follows;

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5}$$

The hypergeometric functions are the solutions of the second order ordinary hypergeometric differential equations throughout the complex plane. Barne [2] systematized the generalized hypergeometric function defined as

$${}_pF_q\left[\begin{array}{c} a_1, ...a_p; z \\ b_1, ...b_q; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k, ...(a_p)_k}{(b_1)_k, ...(b_q)_k} \frac{z^k}{k!} \tag{6}$$

where p and q are integers. If $p = 1$ and $q = 1$ then (6) reduces to

$${}_1F_1\left[\begin{array}{cc} a; & z \\ b; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{7}$$

for $b \neq 0$, and a a non-negative integer. The equation (7) is convergent for all finite values of z. It is also called the Pochhammer-Barnes confluent hypergeometric function [13]. The confluent hypergeometric equations are known as Kummer's equations. According to Mathews [5], the equation (7) can be written as

$${}_1F_1(a; b; z) = \phi(a; b; z) = M(a; b; z) \tag{8}$$

The solution $M(a; b; z)$ in terms of power series, is called the Confluent Hypergeometric Function of the first kind. These functions are widely used in Landan levels [7] and Boundary value problems in Coulomb problem for Hydrogen [3]. The equation (8) is the solution of the differential equation

$$\theta(\theta + b - 1) - z(\theta + a)]w = 0, \tag{9}$$

For $\theta = z\frac{d}{dz}$. The equation (9) can be written as [11]

$$zw'' + (b - z)w' - aw = 0 \tag{10}$$

If the value of b is the non-integral, then the solution of (10) is given by

$$w = A.{}_1F_1(a; b; z) + Bz^{1-b}.{}_1F_1(a + 1 - b; 2 - b; z) \tag{11}$$

where A and B are arbitrary constants.

1.2 Kummer's formulas [11]

The Kummer's First formula is given by

$$e^{-z}.{}_1F_1(a; b; z) = \sum_{n=0}^{\infty}[\frac{(b-a)_n(-z)^n}{(b)_n n!}] \tag{12}$$

and the Kummer's second formula is

$$e^{-z}{}_1F_1(a; 2a; 2z) ={}_0 F_1\left[\begin{array}{cc} -; & \frac{1}{4}z^2 \\ a + \frac{1}{2}; & \end{array}\right] \tag{13}$$

1.3 The solutions at the singularities

A hypergeometric differential equation [11] is given by

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0 \tag{14}$$

The equation (14) has a regular singularity at $z = 0, 1$ and ∞. [15, 17]. The corresponding exponent pairs are thus $(0, 1-c), (0, c-a-b)$. The complex values for a and c may exist [8]. The fundamental solutions for the real values of a and c are given below.

1.3.1 For singularity at $z = 0$,

$$f_1(z) = F\left[\begin{matrix} a, & b; & 1-z \\ & c; & \end{matrix}\right] \tag{15}$$

and

$$f_2(z) = z^{1-c}F\left[\begin{matrix} a-c+1, & b-c+1; & z \\ & 2-c; & \end{matrix}\right] \tag{16}$$

such that

$$W[f_1(z), f_2(z)] = (1-c)z^{-a}(1-z)^{c-a-b-1} \tag{17}$$

1.3.2 For singularity at $z = 1$,

$$f_1(z) = F\left[\begin{matrix} a, & b; & z \\ & c; & \end{matrix}\right] \tag{18}$$

and

$$f_2(z) = z^{c-a-b}F\left[\begin{matrix} c-a, & c-b; & 1-z \\ & c-a-b+1; & \end{matrix}\right] \tag{19}$$

such that

$$W[f_1(z), f_2(z)] = (a+b-c)z^{-c}(1-z)^{-a-b-1} \tag{20}$$

1.3.3 For singularity at $z = \infty$,

$$f_1(z) = F\left[\begin{matrix} a, & b; & 1-z \\ & a+b+1-c; & \end{matrix}\right] \tag{21}$$

and

$$f_2(z) = z^{1-c}F\left[\begin{matrix} a-c+1, & b-c+1; & z \\ & 2-c; & \end{matrix}\right] \tag{22}$$

such that

$$W[f_1(z), f_2(z)] = (1-b)z^{-c}(z-1)^{c-a-b-1} \tag{23}$$

The irregular singularities of the confluent hypergeometric functions are exactly the same as the exponential functions, obtained by using Gauss power series expansion[10].

1.4 Kummer's formula for ${}_2F_1$ [1]

The Kummer's formula for ${}_2F_1(a; b; 1+a-b; z)$ is

$${}_2F_1(a; b; 1+a-b; z) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \tag{24}$$

1.5 Kummer's solutions and connecting formulas

1.5.1 Basic solutions [12]

The solutions (15)-(23) can be transformed into six other basic solutions through the equation

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1\left[\begin{array}{ccc} c-a, & c-b; & z \\ & c; & \end{array}\right] \tag{25}$$

The equation (25) will give six solutions which are known as the Kummers solutions [12]. Morita et al. [6] has obtained the solutions to Kummer's solution through fractional calculus. The six solutions obtained by Kummer [17] are as follows

$$ w_1(z) = (1-z)^{-b}{}_2F_1\left[\begin{array}{ccc} c-a, & b; & \frac{z}{z-1} \\ & c; & \end{array}\right] \tag{26}$$

$$ w_2(z) = (z)^{1-c}(1-z)^{c-a-b}{}_2F_1\left[\begin{array}{ccc} 1-a, & b-c+1; & \frac{z}{z-1} \\ & 2-c; & \end{array}\right] \tag{27}$$

$$ w_3(z) = z^{-b}{}_2F_1\left[\begin{array}{ccc} b, & b-c+1; & 1-\frac{1}{1-z} \\ & a+b-c+1; & \end{array}\right] \tag{28}$$

$$ w_4(z) = (z)^{b-c}(1-z)^{c-a-b}{}_2F_1\left[\begin{array}{ccc} 1-b, & c-b; & 1-\frac{1}{z} \\ & c-a-b+1; & \end{array}\right] \tag{29}$$

$$ w_5(z) = e^{(c-1)\pi}(z)^{1-c}(1-z)^{c-a-b}{}_2F_1\left[\begin{array}{ccc} 1-b, & a-c+1; & \frac{1}{1-z} \\ & a-b+1; & \end{array}\right] \tag{30}$$

$$ w_6(z) = e^{(c-1)\pi i}(z)^{1-c}(1-z)^{c-b-1}{}_2F_1\left[\begin{array}{ccc} 1-a, & b-c+1; & \frac{1}{1-z} \\ & b-a+1; & \end{array}\right] \tag{31}$$

1.6 Connection formulas

By using the property (5), the six solutions (26)-(31) of the three parameters a, b, c and on combining any three solutions, there are ${}_6C_3 = 20$ connection formulas as the principle branches of Kummer's solution [15, 17].They are as follow

$$ w_3(z) = \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z) + \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}w_2(z) \tag{32}$$

$$ w_4(z) = \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}w_1(z) + \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z) \tag{33}$$

$$ w_5(z) = \frac{\Gamma(1-c)\Gamma(a-b+1)}{\Gamma(a-c+1)\Gamma(1-b)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)}w_2(z) \tag{34}$$

$$ w_6(z) = \frac{\Gamma(1-c)\Gamma(b-a+1)}{\Gamma(b-c+1)\Gamma(1-a)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)}w_2(z) \tag{35}$$

$$ w_1(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}w_4(z) \tag{36}$$

$$w_2(z) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} w_3(z) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} w_4(z) \tag{37}$$

$$w_5(z) = e^{a\pi i}\frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} w_3(z) + e^{(c-b)\pi i}\frac{\Gamma(a-b+1)\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)} w_4(z) \tag{38}$$

$$w_6(z) = e^{b\pi i}\frac{\Gamma(b-a+1)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)} w_3(z) + e^{(c-a)\pi i}\frac{\Gamma(b-a+1)\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)} w_4(z) \tag{39}$$

$$w_1(z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} w_5(z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} w_6(z) \tag{40}$$

$$w_2(z) = e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(b-c+1)} w_5(z) + e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(a-c+1)} w_6(z) \tag{41}$$

$$w_3(z) = e^{-a\pi i}\frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)} w_5(z) + e^{-b\pi i}\frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)} w_6(z) \tag{42}$$

$$w_4(z) = e^{(b-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)} w_5(z) + e^{(a-c)\pi i)}\frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)} w_6(z) \tag{43}$$

$$w_1(z) = e^{b\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)} w_3(z) + e^{(a-c)\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)} w_5(z) \tag{44}$$

$$w_2(z) = e^{a\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)} w_3(z) + e^{(a-c)\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)} w_6(z) \tag{45}$$

$$w_1(z) = e^{(b-c+1)\pi i}\frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a+b-c+1)\Gamma(1-b)} w_3(z) + e^{(b-c)\pi i}\frac{\Gamma(2-c)\Gamma(a)}{\Gamma(a-b+1)\Gamma(b-c+1)} w_5(z) \tag{46}$$

$$w_2(z) = e^{(a-c+1)\pi i}\frac{\Gamma(2-c)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(1-a)} w_3(z) + e^{(a-c)\pi i}\frac{\Gamma(2-c)\Gamma(b)}{\Gamma(b-a+1)\Gamma(a-c+1)} w_6(z) \tag{47}$$

$$w_1(z) = e^{(c-a)\pi i}\frac{\Gamma(c)\Gamma(1-b)}{\Gamma(a)\Gamma(c-a-b+1)} w_4(z) + e^{-a\pi i}\frac{\Gamma(c\Gamma(1-b)}{\Gamma(a-b+1)\Gamma(c-a)} w_5(z) \tag{48}$$

$$w_1(z) = e^{(c-b)\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a-b+1)} w_4(z) + e^{-b\pi i}\frac{\Gamma(c)\Gamma(1-a)}{\Gamma(b-a+1)\Gamma(c-b)} w_6(z) \tag{49}$$

$$w_2(z) = e^{(1-a)\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-c+1)\Gamma(c-a-b+1)} w_4(z) + e^{-a\pi i}\frac{\Gamma(2-c)\Gamma(c-b)}{\Gamma(a-b+1)\Gamma(a-1)} w_5(z) \tag{50}$$

$$w_2(z) = e^{(1-b)\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(c-a-b+1)} w_4(z) + e^{-b\pi i}\frac{\Gamma(2-c)\Gamma(c-a)}{\Gamma(b-a+1)\Gamma(1-b)} w_6(z) \tag{51}$$

2 Main Results

In section 1, we have presented the six solutions and twenty connection formulas previously obtained by Kummer. The relationship among any three solutions is precisely shown in the connection formulas. The relationship between any of these four solutions is not yet established. So the objective of this research paper is to establish the combination relationship between three solutions from the set of above (32)-(51) identities. One of the solutions will be expressed as the linear of the other three sets of solutions. By this process we will evaluate the extension solutions for $w_1(z), w_2(z), w_3(z), w_4(z), w_5(z)$ and $w_6(z)$. The extended solution is evaluated through the method of elimination of the common solution between any two identities.

2.1 Extension of connection formulas

The connection formulas for the three solutions of the hypergeometric differential equations are derived by using the relations in the subsection 1.6. The two equi solutions obtained from the Kummer's solutions are simplified to obtain the linear combination of three solutions. That is a solution is expressed as the linear combination of three solutions. Altogether six solutions are obtained through this process.

From (32) and (42), we get

$$\frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)+\frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}w_2(z)=e^{-a\pi i}\frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}w_6(z)$$

or, $\frac{\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)+\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}w_2(z)=e^{-a\pi i}\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}w_6(z)$

or, $\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}w_2(z)=e^{-a\pi i}\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)}w_6(z)-\frac{\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)$

or,

$$w_1(z)=\frac{1}{\Gamma(1-c)}[e^{(b-c)\pi i}\frac{\Gamma(b-a)\Gamma(1-b)}{\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i}\frac{\Gamma(a-b)\Gamma(1-a)}{\Gamma(c-b)}w_6(z) \\ -\frac{\Gamma(c-1)\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)] \tag{52}$$

From (33) and (43), we get

$$\frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)=e^{(b-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i)}\frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)$$

or, $\frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)=e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i)}\frac{\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)$

or, $\frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)=e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i)}\frac{\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)-\frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)$

or,

$$w_2(z)=\frac{1}{\Gamma(c-1)}[e^{-a\pi i}\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b-c+1)}w_5(z)+e^{-b\pi i}\frac{\Gamma(a-b)\Gamma(b)}{\Gamma(a-c+1)}w_6(z) \\ -\frac{\Gamma(1-c)\Gamma(a)\Gamma(b)}{\Gamma(a-c+1)\Gamma(b-c+1)}w_1(z)] \tag{53}$$

From (34) and (38) we get,

$\frac{\Gamma(1-c)\Gamma(a-b+1)}{\Gamma(a-c+1)\Gamma(1-b)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)}w_2(z) = e^{a\pi i}\frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}w_3(z)+e^{(c-b)\pi i}\frac{\Gamma(a-b+1)\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)}w_4(z)$

or, $e^{a\pi i}\frac{\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}w_3(z) + e^{(c-b)\pi i}\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)}w_4(z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(1-b)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)}w_2(z)$

or, $e^{a\pi i}\frac{\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}w_3(z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(1-b)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)}w_2(z) - e^{(c-b)\pi i}\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(a-c+1)}w_4(z)$

$$\begin{aligned} w_3(z) = \frac{e^{-a\pi i}}{\Gamma(c-a-b)}[&\frac{\gamma(1-c)\Gamma(c-b)}{\Gamma(a-c+1)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(1-b)}{\Gamma(a)}w_2(z) \\ &-e^{(c-b)\pi i}\frac{\Gamma(a+b-1)\Gamma(1-b)\Gamma(c-b)}{\Gamma(a)\Gamma(a-c+1)}w_4(z)] \end{aligned} \tag{54}$$

From (35) and (39), we get

$\frac{\Gamma(1-c)\Gamma(b-a+1)}{\Gamma(b-c+1)\Gamma(1-a)}w_1(z)+e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)}w_2(z) = e^{b\pi i}\frac{\Gamma(b-a+1)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)}w_3(z)+e^{(c-a)\pi i}\frac{\Gamma(b-a+1)\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)}w_4(z)$

or, $e^{b\pi i}\frac{\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)}w_3(z) + e^{(c-a)\pi i}\frac{\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)}w_4(z) = \frac{\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(1-a)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)}{\Gamma(b)\Gamma(c-a)}w_2(z)$

or, $e^{(c-a)\pi i}\frac{\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)}w_4(z) = \frac{\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(1-a)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)}{\Gamma(b)\Gamma(c-a)}w_2(z) - e^{b\pi i}\frac{\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)}w_3(z)$

or,

$$\begin{aligned} w_4(z) = \frac{e^{(a-c)\pi i}}{\Gamma(a+b-c)}[&\frac{\Gamma(1-c)\Gamma(b)}{\Gamma(1-a)}w_1(z) + e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-c+1)}{\Gamma(c-a)}w_2(z) \\ &-e^{b\pi i}\frac{\Gamma(c-a-b)\Gamma(b)\Gamma(c-b+1)}{\Gamma(1-a)\Gamma(c-a)}w_3(z)] \end{aligned} \tag{55}$$

From (36) and (40), we get

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}w_4(z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}w_5(z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}w_6(z)$$

or,

$$\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}w_5(z) = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}w_3(z) + \frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}w_4(z) - \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}w_6(z)$$

or,

$$w_5(z) = \frac{1}{\Gamma(b-a)}[\frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-b)}w_3(z) + \frac{\Gamma(a+b-c)\Gamma(c-a)}{\Gamma(a)}w_4(z) - \frac{\Gamma(a-b)\Gamma(b)\Gamma(c-a)}{\Gamma(a)\Gamma(c-b)}w_6(z)] \tag{56}$$

From (32) and (43), we get

$\frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}w_1(z)+\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z) = e^{(b-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)+e^{(a-c)\pi i)}\frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z)$

or, $e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z) + e^{(a-c)\pi i)}\frac{\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z) = \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z) + \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z)$

or, $e^{(a-c)\pi i)}\frac{\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)}w_6(z) = \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z) + \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z) - e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)$

or,

$$w_6(z) = e^{(c-a)\pi i)}\frac{\Gamma(1-b)\Gamma(c-b)}{\Gamma(a-b)}[\frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}w_1(z) + \frac{\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}w_2(z) \\ -e^{(b-c)\pi i}\frac{\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)}w_5(z)] \quad (57)$$

3 Conclusion

The hypergeometric differential equation of second order has a solution in terms of hypergeometric function. The confluent hypergeometric function ${}_1F_1(a;b;z)$ can be evaluated through the novel integration technique which arrives in wave propagation problems [4]. As the solution of the hypergeometric differential equation, Kummer has obtained six solutions (26)-(31) and twenty connection formulas (32)-(51). By the help of these connection formulas, we have obtained six extensions formulas (52)-(57) for $w_1(z), w_2(z), w_3(z), w_4(z), w_5(z)$ and $w_6(z)$. Every extension formulas are expressed as the linear combination of other three solutions. These formulas are highly applicable in physics, social sciences, chemistry and engineering.

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