On Certain Type of Difference Sequence Spaces Defined by Φ-Function

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Abstract: *So far a large number of research works have been studied and investigated in basic sequence spaces c_0, c, and l_∞. In this present work, we introduce the difference sequence spaces $W_0(\Delta, f), W(\Delta, f)$ and $W_\infty(\Delta, f)$ defined by non-negative real-valued Φ-function on $\mathbb{R}$ and study some of their topological properties defined by the paranormed structure on these spaces.*

Keywords: Sequence spaces, Difference sequence space, Paranormed space, Orlicz function, Normal space

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1 Introduction

The classical sequence space is a special case of function space if the domain is restricted to the set of natural numbers. The vector space of all sequences of complex numbers is denoted by ω. Any linear subspace of ω is called a sequence space. Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences with complex terms respectively are defined by

$$\begin{aligned} l_\infty &= \{x = (x_k) \in \omega : \sup_k |x_k| < \infty\} \\ c &= \{x = (x_k) \in \omega : \exists\, l \in \mathbb{C} \text{ such that } |x_k - l| \to 0 \text{ as } k \to \infty\} \\ c_0 &= \{x = (x_k) \in \omega : |x_k| \to 0 \text{ as } k \to \infty\} \end{aligned}$$

and norm is given by

$$||x|| = \sup_k |x_k|, \quad k \in \mathbb{N}.$$

Definition 1.1. A linear space X is said to be a paranormed space if there is a function $g : X \to \mathbb{R}$ satisfying [22]

1. $g(\theta) = 0$ where $\theta = (0, 0, \cdots)$ be a zero vector in X,
2. $g(x) = g(-x)$,
3. $g(x + y) \le g(x) + g(y)$ (subadditivity), and
4. the scalar multiplication is continuous.

A paranorm g is called total if $g(x) = 0$ if and only if $x = \theta$ [22]. The pair (X, g) is called total paranorm space. Nakano [12] and Simmons [20] introduced the notion of paranormed sequence space. Later on, it was further investigated by some other authors like Maddox [9], Tripathy and Sen [21] and Pahari [13, 14, 15, 16, 17].

Definition 1.2. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ [7].

An Orlicz function M is said to satisfy Δ_2-condition [7] if there exists a constant $L > 0$ such that $M(2x) \leq LM(x)$ for all $x \geq 0$.

W. Orlicz used the idea of Orlicz function to construct the Orlicz sequence space. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the Orlicz sequence space

$$l_M = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left[M\left(\frac{|x_k|}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.$$

The space l_M becomes a Banach space [8] with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

.

The space l_M is called an Orlicz sequence space and is closely related to the sequence space l_p with

$$M(x) = x^p, \quad (1 \leq p < \infty).$$

The various algebraic and topological properties of sequence spaces with the help of Orlicz function have been introduced and studied as a generalization of various sequence spaces. For instance, we refer a few: Bala [1], Erdem and Demiriz [2], Khan [4], Kolk [6], Mishra et. al [10], Parashar and Chaudhary [18] and Rao and Ren [19].

Definition 1.3. A sequence space S is said to be solid (normal) [11] if for any sequence (x_k) in a sequence space X and for all sequences (λ_k) of scalars with $|\lambda_k| \leq 1$ for all natural number k implies that $(\lambda_k x_k) \in X$.

Definition 1.4. For any sequence $x = (x_k)$, the difference sequence Δx is defined by

$$\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k-1})_{k=1}^{\infty}.$$

Kizmaz [5] defined the following three sequence spaces
$l_\infty(\Delta) = \{x \in \omega : \Delta x \in l_\infty\}$,
$c(\Delta) = \{x \in \omega : \Delta x \in c\}$, and
$c_0(\Delta) = \{x \in \omega : \Delta x \in c_0\}$.

A sequence $x = (x_k)$ is called Δ-convergent if the lim x_k is finite and hence exists. Every convergent sequence is Δ-convergent but not conversely. For, consider the sequence $x_k = k+1$ for all natural numbers k. Then, $(\Delta x_k) = (x_k - x_{k+1}) = -1$ for each natural numbers k. Thus, $x = (x_k)$ is divergent but it is Δ-convergent. This example illustrates the importance of studying the difference sequences.

Definition 1.5. A continuous function $f : \mathbb{R} \to [0, \infty)$ is called a Φ -function [3] if $f(t) = 0$ if and only if $t = 0$, even and non- decreasing on $[0, \infty)$. The Φ-function is closely related to the Orlicz function.

Herawati and Gultom [3] in 2019 introduced certain type of sequence spaces defined by Φ-function and studied their paranormed structures on these spaces.

We now introduce the following class of difference sequences by extending the concept of sequence space studied in Herawati and Gultom [3].

- $W_0(\Delta, f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0); \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho} \right) \to 0 \text{ as } n \to \infty \right\}$.
- $W(\Delta, f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0)(\exists l > 0); \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{|\Delta x_k - l|}{\rho} \right) \to 0 \text{ as } n \to \infty \right\}$.
- $W_\infty(\Delta, f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0); sup_n \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho} \right) < \infty \right\}$.

where f is a Φ-function.

2 Main Results

In this section, we shall study some topological properties of the class $W_0(\Delta, f)$.

Theorem 2.1. *If Φ-function f satisfies Δ_2-condition, then $W_0(\Delta, f)$ forms a linear space over $\mathbb{C}$.*

Proof. Let $x = (x_k)$ and $y = (y_k)$ be sequences in $W_0(\Delta, f)$. Then there exist ρ_1 and $\rho_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho_1}\right) \to 0 \text{ as } n \to \infty \tag{1}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta y_k|}{\rho_2}\right) \to 0 \text{ as } n \to \infty. \tag{2}$$

Let us choose $\rho = \max\{\rho_1, \rho_2\}$. Using the non-decreasing property of f on $[0, \infty)$ and in view of (1) and (2), we can write

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k + \Delta y_k|}{\rho}\right) &\leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) + \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta y_k|}{\rho}\right) \\ &\leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho_1}\right) + \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta y_k|}{\rho_2}\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that

$$x + y \in W_0(\Delta, f).$$

Again, let $x \in W_0(\Delta, f)$ and $\alpha \in \mathbb{C}$. Then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) \to 0 \text{ as } n \to \infty.$$

We need to show that $\alpha x \in W_0(\Delta, f)$.
The proof is obvious if $\alpha = 0$. So, let $\alpha \neq 0$. Then $|\alpha| > 0$.
By Archimedian property of real numbers, there exists $n_1 \in \mathbb{N}$ such that $|\alpha| \leq 2^{n_1}$. Since f is non-decreasing function on $[0, \infty)$ and f satisfies Δ_2-condition, there exists $M > 0$, such that

$$f(|\alpha| x_k) \leq f(2^{n_1} x_k) \leq M^{n_1} f(x_k) \text{ for all } k \in \mathbb{N}.$$

Hence,

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha \Delta x_k|}{\rho}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha||\Delta x_k|}{\rho}\right) \leq \frac{M^{n_1}}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) \to 0 \text{ as } n \to \infty.$$

This shows that $\alpha x \in W_0(\Delta, f)$ and hence $W_0(\Delta, f)$ is a linear space. □

Theorem 2.2. *The space $W_0(\Delta, f)$ is a paranormed space with a paranorm*
$g : W_0(\Delta, f) \to \mathbb{R}$ defined by

$$g(x) = inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) \leq 1 \,; n \in \mathbb{N}\right\}.$$

Proof. The proof of $g(x) \geq 0$ and $g(-x) = g(x)$ can be easily shown for all $x \in W_0(\Delta, f)$.
For the third property of paranorm, let $x, y \in W_0(\Delta, f)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho_1}\right) \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta y_k|}{\rho_2}\right) \to 0 \text{ as } n \to \infty.$$

Using non-decreasing property of f on $[0, \infty)$, we can obtain

$$\begin{aligned} g(x+y) &= \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k + \Delta y_k|}{\rho}\right) \leq 1\right\} \\ &\leq \inf\left\{\rho_1 > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho_1}\right) \leq 1\right\} + \inf\left\{\rho_2 > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho_2}\right) \leq 1\right\} \\ &= g(x) + g(y). \end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y) \text{ for all } x, y \in W_0(\Delta, f).$$

Finally, we prove the continuity of scalar multiplication.
Let $x \in W_0(\Delta, f)$ be such that

$$g(x_k^{(n)} - x_k) \to 0 \text{ as } n \to \infty$$

and (α_n) a sequence of scalars such that $\alpha_n \to \alpha$ as $n \to \infty$.
Now,

$$\begin{aligned} g(\alpha_n x_k^{(n)} - \alpha x_k) &= \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha_n \Delta x_k^{(n)} - \alpha_n \Delta x_k|}{\rho}\right) \leq 1\right\} \\ &\leq \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha_n \Delta x_k^{(n)} - \alpha \Delta x_k^{(n)}|}{\rho}\right) \leq 1\right\} \\ &\qquad + \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha \Delta x_k^{(n)} - \alpha \Delta x_k|}{\rho}\right) \leq 1\right\} \\ &= |\alpha_n - \alpha| \inf\left\{\rho_1 = \left(\frac{\rho}{|\alpha_n - \alpha|}\right) > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k^{(n)}|}{\rho_1}\right) \leq 1\right\} \\ &\qquad + |\alpha| \inf\left\{\rho_2 = \left(\frac{\rho}{|\alpha|}\right) > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k^{(n)} - \Delta x_k|}{\rho_2}\right) \leq 1\right\} \\ &= |\alpha_n - \alpha|\, g\left(x_k^{(n)}\right) + |\alpha| g\left(x_k^{(n)} - x_k\right) \\ &\quad \to 0 \text{ as } \alpha_n \to \alpha \text{ and } g\left(x_k^{(n)} - x_k\right) \to 0. \end{aligned}$$

Hence, $W_0(\Delta, f)$ is a paranormed space. □

Theorem 2.3. *If f as Φ-function satisfies the convex and Δ_2-condition, then the space $W_0(\Delta, f)$ is a complete paranormed space.*

Proof. Let $\left(x_k^{(n)}\right)$ be a Cauchy sequence in $W_0(\Delta, f)$, where $\left(x_k^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \cdots\right)$. Then there exists $n_1 \in \mathbb{N}$ such that for every $m \geq n \geq n_1$;

$$g\left(x^{(m)} - x^{(n)}\right) = \epsilon \cdot \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(m)} - \Delta x_k^{(n)}|}{\epsilon}\right) \leq 1.$$

Using the convexity of f, we have

$$\frac{1}{s} \sum_{k=1}^{s} f\left(|\Delta x_k^{(m)} - \Delta x_k^{(n)}|\right) \leq \epsilon \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(m)} - \Delta x_k^{(n)}|}{\epsilon}\right) < \epsilon$$

Since $\epsilon > 0$ was arbitrary,

$$f\left(|\Delta x_k^{(m)} - \Delta x_k^{(n)}|\right) = 0 \text{ for all } m \geq n \geq n_1.$$

This follows that

$$|x_k^{(m)} - x_k^{(n)}| < \epsilon \text{ for all } m \geq n \geq n_1.$$

This shows that $\left(x_k^{(n)}\right)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $x_k \in \mathbb{R}$ such that $\lim_{n\to\infty} x_k^{(n)} = x_k$.
Thus for every $n \geq n_1$,

$$|x_k^{(m)} - x_k| = |x_k^{(m)} - \lim_{n\to\infty} x_k^{(n)}| = \lim_{n\to\infty} |x_k^{(m)} - x_k^{(n)}| < \epsilon^2.$$

Since $\left(x_k^{(n)}\right) \in W_0(\Delta, f)$, we can write

$$\frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)}|}{\rho}\right) \to 0 \text{ as } s \to \infty.$$

Using continuity of f, we have

$$\frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k|}{\rho}\right) = \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\lim_{n\to\infty} \Delta x_k^{(n)}|}{\rho}\right) = \lim_{n\to\infty} \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)}|}{\rho}\right) = 0 \text{ as } s \to \infty.$$

Thus,

$$\frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k|}{\rho}\right) \to 0 \text{ as } s \to \infty.$$

Hence, $(x_k) \in W_0(\Delta, f)$.
Finally, we show that $g(x^{(n)} - x) \to 0$ as $n \to \infty$.
Using the continuity of f, we have

$$\frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)} - \Delta x_k|}{\rho}\right) \leq \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)} - \lim_{m\to\infty} \Delta x_k^{(m)}|}{\rho}\right) = \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)} - \Delta x_k^{(m)}|}{\rho}\right) \leq 1.$$

Thus

$$g(x^{(n)} - x) = \inf\left\{\rho > 0 : \frac{1}{s} \sum_{k=1}^{s} f\left(\frac{|\Delta x_k^{(n)} - \Delta x_k|}{\rho}\right) \leq 1\right\}.$$

This implies that $g(x^{(n)} - x) < \rho$ for every $\rho > 0$.

It follows that there exists a real sequence $\left(\frac{p}{2^q}\right), q \geq 1$ for a real number p, together with

$$g(x^{(n)} - x) < \frac{p}{2^q}, q \geq 1.$$

Thus we obtain $g(x^{(n)} - x) \to 0$ as $n \to \infty$.
Hence, $W_0(\Delta, f)$ is a complete paranormed space.

□

Theorem 2.4. *The space $W_0(\Delta, f)$ is normal.*

Proof. Let $x = (x_k) \in W_0(\Delta, f)$. Then there exists $\rho > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) \to 0 \text{ as } n \to \infty$$

.
Let (α_k) be a sequence of scalars satisfying $|\alpha_k| \leq 1$ for all $k \geq 1$.
Using $|\alpha_k| \leq 1$ for all $k \geq 1$ and non-decreasing property of f, we have

$$f(|\alpha_k|\Delta x_k) \leq f(\Delta x_k)$$

Then,

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha_k \Delta x_k|}{\rho}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha_k||\Delta x_k|}{\rho}\right) \leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\Delta x_k|}{\rho}\right) \to 0 \text{ as } n \to \infty.$$

This shows that $\alpha_k x_k \in W_0(\Delta, f)$ and hence $W_0(\Delta, f)$ is normal.

□

3 Conclusions

Here we established some of the results that characterize the linear topological properties of the difference sequence space $W_0(\Delta, f)$ defined by non-negative real valued Φ-function on $\mathbb{R}$. Moreover, the results can be used to prove the results related to the linear and topological properties of the classes $W(\Delta, f)$ and $W_\infty(\Delta, f)$.

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