On a New Application of Positive and Decreasing Sequences to Double Fourier Series Associated with $(N, p_m^{(1)}, p_n^{(2)})$

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Abstract: *In this paper, we introduce a new application of positive and decreasing sequences to double Fourier series associated with $(N, p_m^{(1)}, p_n^{(2)})$. Further, by considering some suitable conditions for previously known results, we have validated the current findings. This work was motivated by the works of [5] and [12].*

Keywords: Positive and decreasing sequences, Summability, Double Fourier series

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1 Introduction

A series is built on the concept of sequence. As a result, sequence and series are concepts that are related and several authors have studied the sequence. Paudel et al. [12] studied sequence and generalized sequence space. Sahani et al. [13] studied series-to-series transformations and analytical continuations using matrix methods. In this paper, we have studied about the double Fourier series. The double Fourier series associated with the function $\varphi(\alpha, \beta)$ is defined in the following way

$$\sum_{g=0}^{\infty}\sum_{h=0}^{\infty}\gamma_{gh}\{r_{gh}\cos g\alpha\,\cos h\beta + s_{gh}\,\sin h\alpha\,\cos h\beta + t_{gh}\cos g\alpha\,\sin h\beta + q_{gh}\sin g\alpha\,\sin h\beta\} \tag{1}$$

where $\varphi(\alpha, \beta)$ is a Lebesgue integrable mapping in the rectangle $R(-\pi,\ \pi; -\pi,\ \pi)$ and is f period 2π [5, 11] and

$$\gamma_{gh} = \begin{cases} 4^{-1} & \text{for} \quad g = 0,\ h = 0 \\ 2^{-1} & \text{for} \quad g = 0,\ h > 0 \,\text{or}\, g > 0,\ h = 0 \\ 1 \quad \text{for} & g,\ h\ > 0 \end{cases}$$

$$r_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\,\beta)\,\cos g\alpha\,\cos h\beta\ d\alpha\,d\beta \tag{2}$$

$$s_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\,\beta)\ \sin g\alpha\ \cos h\beta\ d\alpha\,d\beta \tag{3}$$

$$t_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\,\beta)\ \cos g\alpha\ \sin h\beta\,d\alpha\,d\beta \tag{4}$$

$$q_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\,\beta)\,\sin g\alpha\,\sin h\beta\,d\alpha\,d\beta \tag{5}$$

Also, let

$$\chi(\alpha,\beta) = \chi_{g,\,h}(\alpha,\beta) = 4^{-1}\{\varphi(g+\alpha,\,h+\beta) + \varphi(g-\alpha,\,h+\beta) + \varphi(g+\alpha,\,h-\beta) + \varphi\,(g-\alpha,\ h-\beta) - 4\varphi(\alpha,\beta)\} \tag{6}$$

Definition 1. *Let* $\left\{p_g^{(1)}\right\}$ *and* $\left\{p_h^{(2)}\right\}$ *be sequences of constants and*

$$P_g^{(1)} = p_0^{(1)} + p_1^{(1)} + p_2^{(1)} \ldots + p_g^{(1)}$$

and

$$P_h^{(2)} = p_0^{(2)} + p_1^{(2)} + p_2^{(2)} + \ldots + p_h^{(2)}.$$

The double Nörlund transform $\{a_{gh}\}$ *is defined in the following way [5, 16]*

$$V_{gh} = \frac{1}{P_g^{(1)} P_h^{(2)}} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} P_{g-1}^{(1)} \; P_{h-1}^{(1)} \; a_{gh}. \tag{7}$$

Definition 2. *If*

$$V_{gh} \to v, \; (g, \; h) \to (\infty, \; \infty) \tag{8}$$

then $\{a_{gh}\}$ *is known as Nörlund sum to a finite sum* v *and is generally denoted by* $(N, \; p_g^{(1)}, \; p_h^{(2)})$ *[5, 16].*

Definition 3. *The Cesàro transform of order is defined by the following way [5, 8, 10, 11, 16]*

$$\begin{array}{l} p_g^{(1)} = 1 \; \forall \; g \geq 0, \\ p_h^{(2)} = 1 \; \forall h \geq 0. \end{array} \tag{9}$$

The Cesàro summability is denoted by (C, 1, 1).

Definition 4. *The summability* $(N, p_g^{(1)}, p_h^{(2)})$ *is said to be harmonic summability [5, 8, 10, 11] if*

$$\begin{array}{r} p_g^{(1)} = (g+1)^{-1} \; \forall \; g \geq 0, \\ \textit{and } p_n^{(2)} = (h+1)^{-1} \; \forall \, h \; \geq 0. \end{array}$$

The conditions for the regularity for Nörlund summability are

$$\sum_{g=0}^{a} |p_g^{(\gamma)}| = O\left(P_g^{(\gamma)}\right), \; \forall g \geq 1 \text{ and } \frac{p_g^{(\gamma)}}{P_g^{(\gamma)}} \to 0 \text{ as } g \to \infty.$$

In the special case in which

$$p_g^{(1)} = (g+1)^{-1} \text{ and } p_h^{(2)} = (h+1)^{-1},$$

$p_g^{(1)} \sim \log g$ and $p_h^{(2)} \sim \log h$ as $g, \; h \to \infty$ respectively, $\{V_{gh}\}$ reduces to the familiar harmonic means of $\{\alpha_{gh}\}$.

2 Main Result

There are several results on summability by Nörlund means of Fourier series. The authors [2, 4, 6, 7, 9, 13, 14] have looked into it. This encourages us to research it in both more generalized and particular cases. To advance, we explore the double Fourier series and its conjugate series using Nörlund means.

Herriot [5] has considered the restricted double Nörlund summability of the rectangular partial sums of the double Fourier series. The goal of our research is to prove the following theorem and Lemmas.

Theorem 1. *Let* $\{p_m^{(1)}\}$ *and* $\{p_n^{(2)}\}$ *be two positive and decreasing sequences such that*

$$\sum_{h=a}^{\zeta} \frac{P_h}{h \log h} = O(P_\zeta),\ \zeta = m \ or\ n$$

and a *is a fixed positive integer and if as* $g, h \to 0$,

$$\begin{aligned} \zeta(g,\, h) &= \int_0^g ds \int_0^h |\chi(s,\, t)|dt = O\left[\frac{g\, h}{\log\frac{1}{g}\ \log\frac{1}{h}}\right], \\ \chi_1(g) &= \int_0^\pi dt| \int_0^g \chi(s,\ t)|ds = O\left[\frac{g}{\log\frac{1}{g}}\right], \\ \chi_1(h) &= \int_0^\pi |ds \int_0^v \chi(s,\ t)|dt = O\left[\frac{h}{\log\frac{1}{h}}\right], \end{aligned}$$

then the given double Fourier series of $\varphi(\alpha,\, \beta)$ *at* $\alpha = g$ *and* $\beta = h$ *is summable by* $(N,\ p_m^{(1)},\ p_n^{(2)})$ *to the sum* $\varphi(g,\ h)$.

3 Few Lemmas

Lemma 1. *If* $\{p_n\}$ *is a positive and decreasing sequence, then for* $0 \le a < b \le \infty,\ 0 \le t \le \pi$ *and* n *and* a, $|\sum_a^b p_k e^{i(n-k)r}| < O(P_\tau)$, *where* $\tau = \frac{1}{t}$ *[16].*

Lemma 2. *For* t *such that* $0 \le t \le \frac{1}{n}$,

$$N_n(t) = \left| \frac{1}{2\pi P_n} \sum_{u=0}^{n} p_u \frac{\sin(n-u+\frac{1}{2})t}{\sin\frac{t}{2}} \right| = O(n),$$

where $N_n(t)$ *is the Nörlund summability kernel for Fourier series [16].*

Proof. We have

$$\begin{aligned} N_n\,(t) &= O\left(\frac{1}{P_n}\sum_{u=0}^{n} p_u \frac{(2n-2u+1)\ \left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|}\right) \\ &= O\left(\frac{2n+1}{P_n}\ \sum_{u=0}^{n} p_u\right) \\ &= O\,(n)\,. \end{aligned}$$

□

Lemma 3. *For each* t *and* $\frac{1}{n} \le t \le \delta$ *[14],*

$$|N_n\ \ (t)| = O\left(\frac{P\left(\frac{1}{t}\right)}{t\, P_n}\right).$$

Proof. Using lemma (1), we may write

$$N_n(t) = \left(\frac{1}{P_n}\left|\sum_{g=0}^{n} P_g \sin(n-g)\, t \; cot\; \frac{t}{2} + \sum_{g=0}^{n} p_g \cos(n-g)\, t\right|\right)$$
$$= O\left(\frac{1}{P_n}\left|\sum_{g=0}^{n} p_g \sin\,(n-g)\, t\;\cot\frac{t}{2}\right|\right) + O\left(\frac{1}{P_n} P\left(\frac{1}{t}\right)\right)$$
$$= O\left(\frac{1}{P_n} P\left(\frac{1}{t}\right)\cot\frac{t}{2})\right) + O\left(\frac{1}{P_n} P\left(\frac{1}{t}\right)\right)$$
$$= O\left(\frac{P\left(\frac{1}{t}\right)}{tP_n}\right).$$

Proof of theorem (1) : Using the result of Harriot [5], we write

$$\pi^2 V_{mn} = \int_0^{\pi}\int_0^{\pi} \chi(g,\ h) N_m^{(i)}(g) N_n^{(2)}(h) dg\, dh$$
$$= \left(\int_0^{\pi}\int_{.}^{\tau} + \int_0^{\delta}\int_{\tau}^{\pi} + \int_{\delta}^{\pi}\int_{\tau}^{\pi}\right)\left\{\chi\,(g,\ h)\, N_m^{(1)}\,(g)\, N_n^{(2)}\,(h)\, dg\ dh\right\}$$

where $\frac{1}{m} < \delta < \pi,\ \frac{1}{n} < \delta < \pi.$

$$= I_1 + I_2 + I_3 + I_4 \tag{10}$$

Now, by Riemann-Lebesgue theorem and regularity method of summation, we have

$$|I_4| = \left|\int_{\delta}^{\pi}\int_{\tau}^{\pi} \chi\,(g,\ h)\, N_m^{(1)}\,(g)\, N_n^{(2)}\,(h)\, dg\ dh\right|$$
$$= O\left(\frac{1}{P_m^{(1)} P_n^{(2)}}\int_{\delta}^{\pi}\int_{\tau}^{\pi}\left|\chi\,(g,\ h)\, N_m^{(1)}\,(g)\, N_n^{(2)}\,(h)\right| dg\, dh\right)$$
$$= O\left(\frac{1}{P_m^{(1)} P_n^{(2)}}\int_{0}^{\pi}\int_{0}^{\pi}\left|\chi\,(g,\ h)\right| dg\, dh\right)\left(\because\quad N_m^{(1)}\,(g)\,,\ N_n^{(2)}\,(h)\ \text{ are even function}\right)$$
$$= o\,(1) \tag{11}$$

$$I_3 = \int_{\delta}^{\pi} N_m^{(1)}\,(g)\ dg\ \int_0^{\pi}\chi\,(g,\ h)\, N_n^{(2)}\,(h)\, dh$$
$$= \int_{\delta}^{\pi} N_m^{(1)}\,(g)\, dg\ \int_0^{\frac{1}{n}}\chi\,(g,\ h)\, N_n^{(2)} dh + \int_{\delta}^{\pi} N_m^{(1)}\,(g)\, dg \int_{\frac{1}{n}}^{\pi}\chi\,(g,\ h)\, N_n^{(2)} dh$$
$$= I_{3,\ 1} + I_{3,\ 2},\quad \text{(say)} \tag{12}$$

Again, by using lemma(2) and lemma (3) and by hypothesis of the theorem, we must write

$$|I_{3,\ 1}| = O\left[\frac{n}{P_m}\int_{\delta}^{\pi} dg \int_0^{\frac{1}{n}}\left|\chi\,(g,\ h)\right| dh\right] = O\,(1)\,. \tag{13}$$

Also,

$$|I_{3,\ 2}| = O\left[\frac{1}{P_m}\int_{\delta}^{\pi} dg \int_{\frac{1}{n}}^{\pi}\left|\chi\,(g,\ h)\right|\frac{P\left(\frac{1}{h}\right)}{h\,P\,(n)}\ dh\right]$$
$$= O\left[\frac{1}{P_m P_n}\int_{\delta}^{\pi} dg \int_{\frac{1}{n}}^{\pi}\left|\chi\,(g,\ h)\right|\frac{P\left(\frac{1}{h}\right)}{h}\ dh\right]$$

By partial integration, we have

$$|I_{3,\,2}| = O\left[\frac{1}{P_m P_n}\chi_1\left(h\right)\frac{P\left(\frac{1}{h}\right)}{h}\right]_{\frac{1}{n}}^{\tau} + O\left[\frac{1}{P_m P_n}\int_{\frac{1}{n}}^{\tau}\chi_1\ \left(h\right)\frac{d}{dh}\left\{\frac{P\left(\frac{1}{h}\right)}{v}\right\}dh\right]$$

$$= O\left[\frac{1}{P_m P_n}\left\{\left(\frac{P\left(\frac{1}{h}\right)}{\log\frac{1}{h}}\right)_{\frac{1}{n}}^{\tau}\right\}\right] + O\left[\frac{1}{P_m P_n}\int_{\frac{1}{n}}^{\tau}\frac{h}{\log\frac{1}{h}}\left|\frac{d}{dh}\left(\frac{P\left(\frac{1}{h}\right)}{h}\right)\right|dh\right]$$

$$= o\left(1\right) + O\left[\frac{1}{P_m P_n}\int_{\tau^{-1}}^{n}\frac{1}{z\log z}\left|\frac{d}{dz}\left\{zp\left(z\right)\right\}\right|dz\right]$$

$$= o\left(1\right) + O\left[\frac{1}{P_m P_n}\int_{\tau^{-1}}^{c}\frac{1}{z\log z}\left|\frac{d}{dz}\left\{zp\left(z\right)\right\}\right|dz\right] + O\left[\frac{1}{P_m P_n}\int_{c}^{n}\frac{1}{z\log z}\left|\frac{d}{dz}\left\{zp\left(z\right)\right\}\right|dz\right],$$

where $c = \left[\tau^{-1}+1\right]$.

$$= o\left(1\right) + O\left(\frac{1}{P_m P_n}\right) + O\left[\frac{1}{P_m P_n}\sum_{c}^{n}\left\{\frac{\Delta h P_h}{h\,\log h}\right\}\right]$$

$$= O\left(\frac{1}{P_m P_n}\right) + O\left[\frac{1}{P_m P_n}\sum_{c}^{n}\frac{P_h}{h\log h}\right] + O\left[\frac{1}{P_m P_n}\ \sum_{c}^{n}\frac{\left(h+1\right)P_n}{h\log h}\right]$$

$$= O\left(\frac{1}{P_m}\right)$$

$$= o\left(1\right),\quad as\ \ P_m\to\infty \tag{14}$$

Similarly, we can show that

$$|I_2| = o\left(1\right) \tag{15}$$

For I_1,

$$I_1 = \left(\int_0^{\frac{1}{m}}\ \int_0^{\frac{1}{n}} + \int_0^{\frac{1}{m}}\ \int_{\frac{1}{n}}^{\tau} + \int_{\frac{1}{m}}^{\delta}\ \int_0^{\frac{1}{n}} + \int_{\frac{1}{m}}^{\delta}\ \int_{\frac{1}{n}}^{\tau}\right)\chi\left(g,\ h\right)N_m^{(1)}\left(g\right)N_n^{(2)}\left(h\right)dg\ dh$$

$$= I_{1,\,1} + I_{1,\,2} + I_{1,\,3} + I_{1,\,4} \tag{16}$$

Again, for $I_{1,\,1}$,

$$|I_{1,1}| = O\left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\chi\left(g,\ h\right)\right|gh\,dg\,dh\right) = O\left(1\right) \tag{17}$$

For $I_{1,\,2}$,

$$|I_{1,\,2}| = O\left[\int_0^{\frac{1}{m}} mdg\ \int_{\frac{1}{n}}^{\tau}\chi\left(g,\ h\right)N_n^{(2)}\left(h\right)dh\right] = O\left(1\right) \tag{18}$$

Similarly,

$$|I_{1,\,3}| = o\left(1\right) \tag{19}$$

For $I_{1,\,4}$,

$$|I_{1,\,4}| = O\left[\int_{\frac{1}{m}}^{\delta}\ \int_{\frac{1}{n}}^{\tau}\left|\chi\left(g,\ \ h\right)\right|\frac{P\left(\frac{1}{g}\right)}{g\,p\left(m\right)}\frac{P\left(\frac{1}{h}\right)}{h\,p\left(n\right)}\ dg\,dh\right]$$

By integrating by parts for double integral, we have

$$|I_{1,\,4}| = O\Bigg[\zeta(\delta,\ \pi)\, P^{(1)}\left(\frac{1}{\delta}\right) \frac{1}{\delta P^{(1)}(m)}\ P^{(1)}\left(\frac{1}{\tau}\right) \frac{1}{\tau P^{(2)}(n)} - \frac{1}{\tau P^{(2)}(n)} P^{(2)}\left(\frac{1}{\tau}\right) \int_{\frac{1}{m}}^{\delta} \zeta(g,\ \tau) \times$$
$$\left|\frac{d}{dg}\left(\frac{P^{(1)}\left(\frac{1}{g}\right)}{g}\right) \frac{1}{P^{(1)}(m)}\right| dg - \frac{1}{\delta P^{(1)}(m)}\ P^{(1)}\left(\frac{1}{\delta}\right) \int_{\frac{1}{n}}^{\tau} \zeta(\delta,\ h) \left|\frac{d}{dh}\left(\frac{P^{(2)}\left(\frac{1}{h}\right)}{h}\right) \frac{1}{P^{(2)}(n)}\right|$$
$$dh \times \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} \zeta(g,\ h) \left|\frac{d}{dg}\left(\frac{P^{(1)}\left(\frac{1}{g}\right)}{g}\right)\right| \left|\frac{1}{P^{(1)}(m)}\right| \frac{d}{dh}\left(\frac{P^{2}\left(\frac{1}{h}\right)}{h}\right)\Bigg| \frac{1}{P^{(2)}(n)}\ dg\ dh$$
$$= J_1 + J_2 + J_3 + J_4 \tag{20}$$

Now,

$$J_1 = o(1) \tag{21}$$
$$J_2 = o(1) \tag{22}$$

As in (21) and (22)

$$J_3 = o(1)\,. \tag{23}$$

Again,

$$J_4 = O\left[\frac{1}{P^{(1)}(m)\ P^{(2)}(n)} \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau} \frac{g}{\log\frac{1}{g}} \left|\frac{d}{dg}\left(\frac{P^{(1)}\left(\frac{1}{g}\right)}{g}\right)\right| \frac{h}{\log\frac{1}{h}} \left|\frac{d}{dh}\left(\frac{P^{(2)}\left(\frac{1}{h}\right)}{h}\right)\right| dg\, dh\right]$$
$$= O\left[\frac{1}{P^{(1)}(m)\, P^{(2)}(n)} \int_{\delta^{-1}}^{m} \frac{1}{z\log z}\left|\frac{d}{dz}\left(zP^{(1)}(z)\right)\right| dz \int_{\tau^{-1}}^{n} \frac{1}{z\log z}\left|\frac{d}{dz}\left(zP^{(2)}(z)\right)\right| dz\right]$$
$$= o(1) \tag{24}$$

Combining (11)-(24), we get (10).
This completes the proof of the theorem. □

Corollary 1. *If we put* $P_m^{(1)} = \dfrac{1}{m+1}$ *and* $P_n^{(2)} = \dfrac{1}{n+1}$ *in the given theorem, we get the result of Sharma [12].*

4 Conclusions

The study of sequence spaces has occupied a very prominent position in analysis. The study of sequence space was motivated by several notable mathematicians, such as Cesàro, Holder, Abel, Nörlund, Euler, Knopp, Hardy and others. In this paper, we have established the least conditions for Nörlund summability of the double Fourier series.

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