Periodic Components of the Fatou Set in Angular Region

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Abstract: *Here we discuss, for a given integer, the existence of transcendental entire function such that its number of periodic Fatou components lie in angular regions and their periodicity are related to the integer.*

Keywords: Fatou set, Fatou component, Periodic Fatou component

DOI: `https://doi.org/10.3126/jnms.v5i1.47373`

1 Introduction

We denote the complex plane by $\mathbb{C}$, extended complex plane which is also known as Riemann sphere by $\mathbb{C}_\infty$ and the set of the natural numbers by $\mathbb{N}$. We assume $D \subset \mathbb{C}_\infty$ and $f : D \to \mathbb{C}_\infty$ is a holomorphic function and $f^n = fofo\dots of(n \text{ times})$ denotes the n^{th} composition of f with itself. If $f^n(z) = z$ for some smallest $n \in \mathbb{N}$, then we say that z is a periodic point of period n. In particular, if $f(z) = z$, then z is a fixed point of f. If $|(f^n)^{'}(z)| < 1$, where $(f^n)'$ represents complex differentiation of f^n with respect to z, then periodic point z is called an attracting periodic point [2].

Let $\mathcal{F}$ be a family of holomorphic functions $f : D \to \mathbb{C}_\infty$. Then $\mathcal{F}$ is a normal family in D if every sequence in $\mathcal{F}$ has a locally uniformly convergent subsequence. The limit function is a holomorphic function or the constant ∞. For $z_0 \in D$, if there exists a neighborhood $N(z_0) \subset D$ of z_0 such that $\mathcal{F}$ is normal in $N(z_0)$, then we say that $\mathcal{F}$ is normal at z_0 [2].

The Fatou and Julia set of f is defined by $F(f) = \{z \in \mathbb{C}_\infty :$ the sequence f^n is well defined and normal at $z\}$ and $J(f) = \mathbb{C}_\infty - F(f)$, respectively [2].

A maximal domain U of normality of iterates is called a component of $F(f)$. For a Fatou component U, if $f^n(U) \subset U$ for some integer $n \geq 1$, then we call U a periodic component of $F(f)$. The minimum n is the period of the component. If $f^m(U)$ is periodic for some integer $m \geq 0$, we call U a pre-periodic component of $F(f)$. Otherwise, all $f^n(U)$ are disjoint, and we call U a wandering domain [2].

Morosawa [3] gave some examples of Baker domains which lie in angular regions. Singh [4] have constructed some examples of periodic and wondering domains of entire functions which lie in angular regions. In particular, they have proved that for any $n \in \mathbb{N}$, there exists a transcendental entire function f, which has a parabolic domain of period n, which lies in an angular region. Tomar [6] studied the existence of wandering and periodic components of Fatou set with all possible combinations under composition of transcendental entire functions in angular region. Subedi and Singh [5] also studied the existence of three different transcendental entire functions and existence of domains which lie in different periodic component of each of these functions and their possible compositions.

In this paper, we extend the result of Singh [4]. That is, we construct a transcendental entire function and its Fatou periodic components for a given integer such that the number of components and their periodicity are related to the integer.

2 Results

Theorem 2.1. *Let n be a positive even integer. Then there exists an entire function f which has n periodic components with period n that lie in angular regions.*

Theorem 2.2. *Let $n \in \mathbb{N}$. Then there exists an entire function f which has $2n$ periodic components with period $2n$ that lie in angular regions.*

3 Proofs of Theorems

We prove the results by using the notion of Carleman set [1] from which we obtain approximation of any holomorphic function by entire functions.

Definition 3.1(Carleman Set [1]) If S is a closed subset of $\mathbb{C}$ and

$$C(S) = \left\{f : S \to \mathbb{C} | f \text{ is a continuous on } S \text{ and analytic in } S^0\right\},$$

then S is called a Carleman set (for $\mathbb{C}$) if for any $g \in C(S)$ and any continuous function ϵ on S, there exists an entire function f such that

$$|g(z) - f(z)| < \epsilon(z) \quad \text{for all } z \in S.$$

The following important result of Carleman set was proved by Nersejan in 1971 [1], (Theorem 4, p. 157).

Theorem 3.1. *Let S be a closed proper subset of $\mathbb{C}$. Then S is a Carleman set in $\mathbb{C}$ if and only if*

1. *the set $\mathbb{C}_\infty - S$ is connected.*
2. *the set $\mathbb{C}_\infty - S$ is locally connected at ∞.*
3. *for every compact subset K of $\mathbb{C}$, there is a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of the interior S^0 of S intersects both K and V.*

3.1 Proof of Theorem 2.1

For the proof of the Theorem (2.1), we construct a Carleman set as follows:

Let $n = 2q$ for some $q \in \mathbb{N}$.

Let $a_k = 10ne^{ik(\frac{\pi}{q})}$, $a_{n+1} = a_1$ $(k = 1, 2, \ldots, n)$.

Define

$$\begin{aligned}
C_0 &= \{z : |z| \leq 1\};\\
L_k &= \left\{z : |z| > 1, \arg(z) = \frac{\arg(a_k) + \arg(a_{k+1})}{2}\right\}, \ k = 1, 2, \ldots, n-1;\\
L_n &= \{z : |z| > 1, \Im(z) = 0\};\\
G_k &= \{z : |z - a_k| \leq 1 \cup \{z : |z| \geq 10n + 1, \arg(z) = \arg(a_k)\}, k = 1, 2, \ldots, n.
\end{aligned}$$

Now define

$$S = C_0 \cup (\cup_{k=1}^{n}(L_k \cup G_k)).$$

Then, by Theorem (3.1), S is a Carleman set in $\mathbb{C}$.
Using continuity of e^{z^m}(for all $m = 1, 2, \ldots, n$), for given $\epsilon = \frac{1}{2}$, it is possible to choose positive numbers δ, δ_{mk} $(k = 1, 2, \ldots, n)$ such that

$$|e^{w^m} - 1| < \frac{1}{2} \text{ whenever } |w^m| < \delta \tag{1}$$

$$|e^{w^m} - (1 + a_{m+k})| < \frac{1}{2} \text{ whenever } |w^m - \log(1 + a_{m+k})| < \delta_{mk}. \tag{2}$$

Define

$$\alpha(z) = \begin{cases} 0 & \text{if } z \in C_0 \cup (\cup_{j=1}^{n} L_k), \\ \log(1 + a_{m+k}) & \text{if } z \in G_{m+k-1}, \quad k = 1, 2, \ldots, n \end{cases}$$

and

$$\epsilon(z) = \begin{cases} \delta & \text{if } z \in C_0 \cup (\cup_{j=1}^{n} L_k), \\ \delta_k & \text{if } z \in G_{m+k-1}, \quad k = 1, 2, \ldots, n. \end{cases}$$

Then, $\alpha(z)$ is continuous on S and analytic in S^0. Hence, there exists an entire function $\gamma(z)$ such that $|\gamma(z) - \alpha(z)| < \epsilon(z)$ for all $z \in S$. Thus,

$$|\gamma(z)| < \delta \quad \text{if} \quad z \in C_0 \cup (\cup_{k=1}^{n} L_k) \tag{3}$$

and

$$|\gamma(z) - \log(1 + a_{m+k})| < \delta_{mk} \quad \text{if} \quad z \in G_{m+k-1}, \quad k = 1, 2, \ldots, n. \tag{4}$$

Therefore, from (2) and (4)

$$|e^{\gamma(z)} - (1 + a_{m+k})| < \frac{1}{2} \quad \text{if} \quad z \in G_{m+k-1}, \quad k = 1, 2, \ldots, n.$$

Now, let $f(z) = e^{\gamma(z)} - 1$, then $f(z)$ is a transcendental entire function. Also,

$$|f(z)| < \frac{1}{2} \quad \text{if} \quad z \in C_0 \cup (\cup_{k=1}^{n} L_k)$$

and (for all $m = 1, 2, \ldots, n$)

$$|f(z) - a_{m+k}| < \frac{1}{2} \quad \text{if} \quad z \in G_{m+k-1}, \quad k = 1, 2, \ldots, n.$$

Thus, f maps $C_0 \cup (\cup_{k=1}^{n} L_k)$ into a smaller disk of C_0 and hence C_0 contains an attracting periodic point ξ such that $f^p(C_0 \cup (\cup_{k=1}^{n} L_k)) \longrightarrow \xi$ as $p \longrightarrow \infty$. Thus, $C_0 \cup (\cup_{k=1}^{n} L_k)$ lies in the Fatou set of f.

Also, each G_{m+k-1} is mapped into the smaller disk $|z - a_{m+k}| < \frac{1}{2}$ by the entire function f. So f maps each G_{m+k-1} into the interior of G_{m+k} and f^n maps G_{m+k-1} into a smaller disk in the interior of itself and consequently contains an attracting periodic point and so, belongs to the Fatou set of f. So, we can construct n domains with period n. That is, for all $p = 1, 2, \ldots, n$, $f^n(G_p) \subset G_p$, each containing an attracting periodic point.

Further, the Fatou component containing $C_0 \cup (\cup_{k=1}^{n} L_k)$ is disjoint from each Fatou component containing G_p. Thus, if D_p is the unbounded Fatou component containing G_p, then since the unbounded Fatou components of transcendental entire functions are simply connected, it follows from the classification of periodic components [2] that for each p with $1 < p \leq n$, D_p is the periodic component of period n which lies in the angular domain bounded by L_{p-1}, L_p and the circular arc

$$z = re^{i\theta}, \left(0 \leq \frac{\arg\, a_{p-1} + \arg\, a_p}{2} \leq \theta \leq \frac{\arg\, a_p + \arg\, a_{p+1}}{2}\right)$$

and D_1 is the periodic component of period n which lies in the angular domain bounded by L_1, L_n and the circular arc $\left(0 \leq \theta \leq \dfrac{\arg\, a_1 + \arg\, a_2}{2}\right)$.

3.2 Proof of Theorem 2.2

For the proof of the Theorem (2.2), we construct a Carleman set as follows:
Let $a_k = 10ne^{ik(\frac{\pi}{n})}$, $a_{2n+1} = a_1$ $(k = 1, 2, \ldots, 2n)$.
Define

$$\begin{aligned} C_0 &= \{z : |z| \leq 1\}; \\ L_k &= \{z : |z| > 1, \arg(z) = \frac{\arg(a_k) + \arg(a_{k+1})}{2}\}, \ k = 1, 2, \ldots, 2n - 1; \\ L_{2n} &= \{z : |z| > 1, \Im(z) = 0\}; \\ G_k &= \{z : |z - a_k| \leq 1 \cup \{z : |z| \geq 10n + 1, \arg(z) = \arg(a_k)\}, k = 1, 2, \ldots, 2n. \end{aligned}$$

Now define,

$$S = C_0 \cup (\cup_{k=1}^{2n}(L_k \cup G_k)).$$

Then, by Theorem (3.1), S is a Carleman set in $\mathbb{C}$.
Using continuity of e^{z^m} (for all $m = 1, 2, \ldots, 2n$) for given $\epsilon = \frac{1}{2}$, it is possible to choose positive numbers δ, δ_{mk} $(k = 1, 2, \ldots, 2n)$ such that

$$|e^{w^m} - 1| < \frac{1}{2} \quad \text{whenever} \quad |w| < \delta \tag{5}$$

$$|e^{w^m} - (1 + a_{m+k})| < \frac{1}{2} \quad \text{whenever} \quad |w^m - \log(1 + a_{m+k})| < \delta_{mk}. \tag{6}$$

Define

$$\alpha(z) = \begin{cases} 0 & \text{if } z \in C_0 \cup (\cup_{j=1}^{2n} L_k), \\ \log(1 + a_{m+k}) & \text{if } z \in G_k, k = 1, 2, \ldots, 2n \end{cases}$$

and

$$\epsilon(z) = \begin{cases} \delta & \text{if } z \in C_0 \cup (\cup_{j=1}^{2n} L_k), \\ \delta_{mk} & \text{if z } \in G_{m+k-1}, k = 1, 2, \ldots, 2n. \end{cases}$$

Then, $\alpha(z)$ is continuous on S and analytic in S^0. Hence, there exists an entire function $\gamma(z)$ such that $|\gamma(z) - \alpha(z)| < \epsilon(z)$ for all $z \in S$. Thus,

$$|\gamma(z)| < \delta \text{ if } z \in C_0 \cup (\cup_{k=1}^{2n} L_k) \tag{7}$$

and

$$|\gamma(z) - \log(1 + a_{m+k})| < \delta_{mk} \text{ if } z \in G_{m+k-1},\ k = 1, 2, \ldots, 2n. \tag{8}$$

Therefore, from (5) and (7)

$$|e^{\gamma(z)} - 1| < \frac{1}{2} \text{ if } z \in C_0 \cup (\cup_{k=1}^{2n} L_k)$$

and from (6) and (8)

$$|e^{\gamma(z)} - (1 + a_{m+k})| < \frac{1}{2} \text{ if } z \in G_{m+k-1},\ k = 1, 2, \ldots, 2n.$$

Now, let $f(z) = e^{\gamma(z)} - 1$, then $f(z)$ is a transcendental entire function. Also,

$$|f(z)| < \frac{1}{2} \text{ if } z \in C_0 \cup (\cup_{k=1}^{2n} L_k)$$

and (for all $m = 1, 2, \ldots, 2n$),

$$|f(z) - a_{m+k}| < \frac{1}{2} \text{ if } z \in G_{m+k-1}, \quad k = 1, 2, \ldots, 2n.$$

Thus, f maps $C_0 \cup (\cup_{k=1}^{2n} L_k)$ into a smaller disk of C_0 and hence C_0 contains an attracting periodic point ξ such that $f^p(C_0 \cup (\cup_{k=1}^{n} L_k)) \longrightarrow \xi$ as $p \longrightarrow \infty$. Thus, $C_0 \cup (\cup_{k=1}^{2n} L_k)$ lies in the Fatou set of f.

Also, each G_{m+k-1} is mapped into the smaller disk $|z - a_{m+k}| < \frac{1}{2}$ by the entire function f. So f

maps each G_{m+k-1} into the interior of G_{m+k} and f^n maps G_{m+k-1} into a smaller disk in the interior of itself, and consequently contains an attracting periodic point and so, belongs to the Fatou set of f. So we can construct $2n$ domains lying in the Fatou set with each $p = 1, 2, \ldots, 2n$, $f^p(G_p) \subset G_p$, containing an attracting periodic point.

Further, the Fatou component containing $C_0 \cup (\cup_{k=1}^{2n} L_k)$ is disjoint from the Fatou component containing G_p for all $p = 1, 2, \ldots, 2n$. Thus, if D_p is the unbounded Fatou Component containing G_p, then since the unbounded Fatou components of transcendental entire functions are simply connected, it follows from the classification of periodic components [2] that each p with $1 < p \leq 2n$, D_p is the periodic component of period $2n$ which lies in the angular domain bounded by L_{p-1}, L_p and the circular arc

$$z = re^{i\theta}, \left(0 \leq \frac{\arg\, a_{p-1} + \arg\, a_p}{2} \leq \theta \leq \frac{\arg\, a_p + \arg\, a_{p+1}}{2}\right)$$

and D_1 is the periodic component of period $2n$ which lies in the angular domain bounded by L_1, L_{2n} and the circular arc $\left(0 \leq \theta \leq \dfrac{\arg\, a_1 + \arg\, a_2}{2}\right)$.

4 Conclusions

We conclude that for any positive even integer n, there exists an entire function f which has n periodic components with period n that lie in angular regions. Also, we conclude that for any $n \in \mathbb{N}$, there exists an entire function f which has $2n$ periodic components with period $2n$ that lie in angular regions.

Our focus in future will be “What about the existence of a general transcendental meromorphic function and its components that satisfy Theorem (2.1) and Theorem (2.2) ?”

5 Acknowledgment

Nirmal Gurung acknowledges University Grants Commission (UGC), Nepal for the M.Phil. fellowship (M.Phil-77/78-S & T-5).

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