

β -Change of Riemannian Space and Special Cases of one-form β

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Abstract: *In this paper, we consider β -change of Finsler metric L given by $\bar{L} = f(L, \beta)$, where f is any positively homogeneous function of degree one in L and β . The objectives of the paper are to obtain the β -change of a Riemannian space, projective change of Finsler metric and to discuss the special cases of one-form β .*

Keywords: β -change, Riemannian space, Finsler metric

1 Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space on a differentiable manifold M^n , equipped with the fundamental function $L(x, y)$. In 2012, Shukla et al. [8] introduced the transformation of Finsler metric called Matsumoto change of metric given by

$$\bar{L}(x, y) = \frac{L^2}{L - \beta}$$

where $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n . They obtained the necessary and sufficient condition for this change of Finsler metric to be projective change.

Prasad et al. [7] introduced an exponential change of Finsler metric given by

$$\bar{L}(x, y) = Le^{\beta/L}$$

and they dealt with the imbedding class numbers of the tangent Riemannian space of corresponding spaces.

Recently, Prasad and Kumari [6] introduced the β -change of Finsler space given by

$$\bar{L}(x, y) = f(L, \beta), \tag{1.1}$$

where f is positively homogeneous function of y^i of degree one in L and β . They also dealt with the imbedding classes of the tangent Riemannian spaces. We have considered the same β -change given by the equation (1.1) and obtained the necessary and sufficient condition under which this change becomes a projective change. The particular cases when the vector field b_i in β is special one have been discussed. The Berwald's connection coefficients for the β -changed space have been calculated.

2 Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space equipped with the fundamental function $L(x, y)$ on the smooth manifold M^n . Let $\beta = b_i(x)y^i$ be a one-form on the manifold M^n , then $L \rightarrow f(L, \beta)$ is the β -change of Finsler metric. If we write $\bar{L} = f(L, \beta)$, where f is any positively homogeneous function of degree one in L and β and $\bar{F}^n = (M^n, \bar{L})$, then the Finsler space \bar{F}^n is said to be obtained from F^n by β -change. The quantities corresponding to \bar{F}^n are denoted by putting bar on those quantities.

The fundamental metric tensor g_{ij} , the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad l_i = \frac{\partial L}{\partial y^i} \quad \text{and} \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \quad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \quad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then,

$$L_i = l_i, \quad L^{-1} h_{ij} = L_{ij}.$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0,$$

where $G^i(x, y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \quad F = \frac{L^2}{2} \tag{2.1}$$

where g^{ij} are the inverse of g_{ij} . The well known Berwald connection $B\Gamma = (G_{jk}^i, G_j^i)$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by [8]

$$G_j^i = \dot{\partial}_j G^i, \quad G_{jk}^i = \dot{\partial}_k G_j^i.$$

The Cartan's connection $C\Gamma = (F_{jk}^i, G_j^i, C_{jk}^i)$ is constructed from the metric function L by the following five axioms [8]:

$$(i) \quad g_{ij|k} = 0 \quad (ii) \quad g_{ij}|_k = 0 \quad (iii) \quad F_{jk}^i = F_{kj}^i \quad (iv) \quad F_{0k}^i = G_k^i \quad (v) \quad C_{jk}^i = C_{kj}^i,$$

where $|_k$ and $|_k$ denote h - and v -covariant derivatives with respect to $C\Gamma$. It is clear that the h -covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the h -covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad 2s_{ij} = b_{i|j} - b_{j|i}. \tag{2.2}$$

3 The β -change of Finsler Metric

The β -change of Finsler metric is given by

$$\bar{L}(x, y) = f(L, \beta), \tag{3.1}$$

where f is positively homogeneous function of degree one in L and β . Homogeneity of f gives

$$L f_1 + \beta f_2 = f, \tag{3.2}$$

where subscripts '1' and '2' denote the partial derivatives with respect to L and β respectively.

Differentiating (3.2) with respect to L and β respectively, we get

$$L f_{11} + \beta f_{12} = 0 \quad \text{and} \quad L f_{12} + \beta f_{22} = 0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{\beta L} = \frac{f_{22}}{L^2},$$

which gives

$$f_{11} = \beta^2 \omega, \quad f_{22} = L^2 \omega, \quad f_{12} = -\beta L \omega, \tag{3.3}$$

where Weierstrass function ω is positively homogeneous function of degree -3 in L and β . Therefore,

$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \quad (3.4)$$

Again, ω_2 is positively homogeneous of degree -4 in L and β , so

$$L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0.$$

Throughout the paper we frequently use the equations (3.2), (3.3) and (3.4) without quoting them. Also, we have assumed that f is not linear function of L and β so that $\omega \neq 0$.

We may put

$$\bar{G}^i = G^i + D^i. \quad (3.5)$$

Then, $\bar{G}_j^i = G_j^i + D_j^i$ and $\bar{G}_{jk}^i = G_{jk}^i + D_{jk}^i$, where $D_j^i = \dot{\partial}_j D^i$ and $D_{jk}^i = \dot{\partial}_k D_j^i$. The tensors D^i , D_j^i and D_{jk}^i are positively homogeneous in y^i of degree two, one and zero respectively. Therefore, we have

$$D_{jk}^i y^k = D_j^i, \quad D_j^i y^j = 2D^i.$$

To find difference tensor D^i , we deal with equation [8] $L_{ij|k} = 0$, that means,

$$\partial_k L_{ij} - L_{ijr} G_k^r - L_{rj} F_{ik}^r - L_{ir} F_{jk}^r = 0. \quad (3.6)$$

Since $\dot{\partial}_i \beta = b_i$, from (3.1), we have

$$\begin{aligned} (a) \quad \bar{L}_i &= f_1 L_i + f_2 b_i, \\ (b) \quad \bar{L}_{ij} &= f_1 L_{ij} + \beta^2 \omega L_i L_j - \beta L \omega (L_i b_j + L_j b_i) + L^2 \omega b_i b_j, \\ (c) \quad \partial_j \bar{L}_i &= f_1 \partial_j L_i + (\beta^2 \omega L_i - \beta L \omega b_i) \partial_j L + (L^2 \omega b_i - \beta L \omega L_i) \partial_j \beta + f_2 \partial_j b_i, \\ (d) \quad \partial_k \bar{L}_{ij} &= f_1 \partial_k L_{ij} + \{\beta^2 \omega L_{ij} + \beta^2 \omega_1 L_i L_j - (\beta L \omega_1 + \beta \omega)(L_i b_j + L_j b_i) + \\ &\quad (2L \omega + L^2 \omega_1) b_i b_j\} \partial_k L + \{-\beta L \omega L_{ij} + (2\beta \omega + \beta^2 \omega_2) L_i L_j - \\ &\quad (L \omega_1 + \beta L \omega_2)(L_i b_j + L_j b_i) + L^2 \omega_2 b_i b_j\} \partial_k \beta + \\ &\quad (\beta^2 \omega L_j - \beta L \omega b_j) \partial_k L_i + (\beta^2 \omega L_i - \beta L \omega b_i) \partial_k L_j - \\ &\quad (\beta L \omega L_j - L^2 \omega b_j) \partial_k b_i - (\beta L \omega L_i - L^2 \omega b_i) \partial_k b_j, \\ (e) \quad \bar{L}_{ijk} &= f_1 L_{ijk} + \beta^2 \omega (L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) - \beta L \omega (b_i L_{jk} + b_j L_{ik} + \\ &\quad b_k L_{ij}) + (\beta^2 \omega_2 + 2\beta \omega)(L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) - \\ &\quad (\beta L \omega_2 + L \omega)(b_i b_j L_k + b_j b_k L_i + b_i b_k L_j) + \beta^2 \omega_1 L_i L_j L_k + L^2 \omega_2 b_i b_j b_k. \end{aligned} \quad (3.7)$$

Since $\bar{L}_{ij|k} = 0$ in \bar{F}^n , after using (3.5), we have

$$\partial_k \bar{L}_{ij} - \bar{L}_{ijr} (G_k^r + D_k^r) - \bar{L}_{rj} (F_{ik}^r + {}^c D_{ik}^r) - \bar{L}_{ir} (F_{jk}^r + {}^c D_{jk}^r) = 0, \quad (3.8)$$

where $\bar{F}_{jk}^i - F_{jk}^i = {}^c D_{jk}^i$.

Substituting in the equation (3.8) the values of $\partial_k \bar{L}_{ij}$, \bar{L}_{ir} and \bar{L}_{ijr} from (3.7)(b),(d),(e) and using (3.6),

we have

$$\begin{aligned}
 & - f_1 \{L_{ijr} D_k^r + L_{rj} D_{ik}^r + L_{ir} D_{jk}^r\} + \{\beta^2 \omega L_{ij} + \beta^2 \omega_1 L_i L_j - \\
 & (\beta L \omega_1 + \beta \omega)(L_i b_j + L_j b_i) + (2L \omega + L^2 \omega_1) b_i b_j\} L_r G_k^r + \\
 & \{-\beta L \omega L_{ij} + (2\beta \omega + \beta^2 \omega_2) L_i L_j - (L \omega + \beta L \omega_2) \times \\
 & (L_i b_j + L_j b_i) + L^2 \omega_2 b_i b_j\} (r_{0k} + s_{0k} + b_r G_k^r) + (\beta^2 \omega L_j - L \beta \omega b_j) \times \\
 & (L_{ir} G_k^r + L_r F_{ik}^r) + (\beta^2 \omega L_i - L \beta \omega b_i)(L_{jr} G_k^r + L_r F_{jk}^r) - \\
 & (L \beta \omega L_j - L^2 \omega b_j)(r_{ik} + s_{ik} + b_r F_{ik}^r) - \\
 & (L \beta \omega L_i - L^2 \omega b_i)(r_{jk} + s_{jk} + b_r F_{jk}^r) - \\
 & \{\beta^2 \omega (L_i L_{jr} + L_j L_{ri} + L_r L_{ij}) - L \beta \omega (b_i L_{jr} + b_j L_{ir} + b_r L_{ij}) + \\
 & (\beta^2 \omega_2 + 2\beta \omega)(L_i L_j b_r + L_i L_r b_j + L_j L_r b_i) - \\
 & (L \beta \omega_2 + L \omega)(L_r b_i b_j + L_i b_j b_r + L_j b_i b_r) + \\
 & \beta^2 \omega_1 L_i L_j L_r + L^2 \omega_2 b_i b_j b_r\} (G_k^r + D_k^r) - \\
 & \{\beta^2 \omega L_r L_j - L \beta \omega (L_r b_j + L_j b_r) + L^2 \omega b_r b_j\} (F_{ik}^r + {}^c D_{ik}^r) - \\
 & \{\beta^2 \omega L_r L_i - L \beta \omega (L_r b_i + L_i b_r) + L^2 \omega b_r b_i\} (F_{jk}^r + {}^c D_{jk}^r) = 0,
 \end{aligned} \tag{3.9}$$

where $\partial_k L = L_r G_k^r$, $\partial_k \beta = r_{0k} + s_{0k} + b_r G_k^r$, $\partial_k L_i = L_{ir} G_k^r + L_r F_{ik}^r$ and $\partial_k b_i = r_{ik} + s_{ik} + b_r F_{ik}^r$.

Contracting (3.9) with y^k , and using the fact that $D_{jk}^i y^j = {}^c D_{jk}^i y^j = D_k^i [2]$, we get

$$\begin{aligned}
 & 2\{f_1 L_{ijr} + \beta^2 \omega (L_i L_{jr} + L_j L_{ri} + L_r L_{ij}) - \\
 & L \beta \omega (b_i L_{jr} + b_j L_{ir} + b_r L_{ij}) + (\beta^2 \omega_2 + 2\beta \omega)(L_i L_j b_r + \\
 & L_i L_r b_j + L_j L_r b_i) - (L \beta \omega_2 + L \omega)(L_r b_i b_j + L_i b_j b_r + L_j b_i b_r) + \\
 & \beta^2 \omega_1 L_i L_j L_r + L^2 \omega_2 b_i b_j b_r\} D^r + \{f_1 L_{rj} + \beta^2 \omega L_r L_j - \\
 & L \beta \omega (L_r b_j + L_j b_r) + L^2 \omega b_r b_j\} D_i^r + \{f_1 L_{ir} + \beta^2 \omega L_r L_i - \\
 & L \beta \omega (L_r b_i + L_i b_r) + L^2 \omega b_r b_i\} D_j^r + (L \beta \omega L_j - L^2 \omega b_j) \times \\
 & (r_{i0} + s_{i0}) + (L \beta \omega L_i - L^2 \omega b_i)(r_{j0} + s_{j0}) + \{\beta L \omega L_{ij} - (2\beta \omega + \beta^2 \omega_2) L_i L_j + \\
 & (L \omega + L \beta \omega_2)(L_i b_j + L_j b_i) - L^2 \omega_2 b_i b_j\} r_{00} = 0,
 \end{aligned} \tag{3.10}$$

where '0' stands for contraction with respect to y^i , viz. $r_{0k} = r_{ik} y^i$, $r_{00} = r_{ij} y^i y^j$.

Next, we deal with $\bar{L}_{i|j} = 0$, that is $\partial_j \bar{L}_i - \bar{L}_{ir} \bar{G}_j^r - \bar{L}_r \bar{F}_{ij}^r = 0$, then

$$\partial_j \bar{L}_i - \bar{L}_{ir} (G_j^r + D_j^r) - \bar{L}_r (F_{ij}^r + {}^c D_{ij}^r) = 0. \tag{3.11}$$

Putting the values of $\partial_j \bar{L}_i$, \bar{L}_{ir} and \bar{L}_r from (3.7) in (3.11) and using equation

$$L_{i|j} = \partial_j L_i - L_{ir} G_j^r - L_r F_{ij}^r = 0,$$

and rearranging the terms, we get

$$\begin{aligned}
 f_2 b_{i|j} & = \{f_1 L_{ir} + \beta^2 \omega L_i L_r + L^2 \omega b_i b_r - L \beta \omega (L_i b_r + L_r b_i)\} D_j^r \\
 & + (L \beta \omega L_i - L^2 \omega b_i)(r_{0j} + s_{0j}) + (f_1 L_r + f_2 b_r) D_{ij}^r,
 \end{aligned}$$

which after using (2.2) gives

$$\begin{aligned}
 2f_2 r_{ij} & = \{f_1 L_{ir} + \beta^2 \omega L_i L_r + L^2 \omega b_i b_r - L \beta \omega (L_i b_r + L_r b_i)\} D_j^r + \\
 & \{f_1 L_{jr} + \beta^2 \omega L_j L_r + L^2 \omega b_j b_r - L \beta \omega (L_j b_r + L_r b_j)\} D_i^r + \\
 & (L \beta \omega L_i - L^2 \omega b_i)(r_{0j} + s_{0j}) + (L \beta \omega L_j - L^2 \omega b_j) \times \\
 & (r_{0i} + s_{0i}) + 2(f_1 L_r + f_2 b_r) D_{ij}^r
 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
 2f_2s_{ij} = & \{f_1L_{ir} + \beta^2\omega L_iL_r + L^2\omega b_ib_r - L\beta\omega(L_ib_r + L_rb_i)\} D_j^r - \\
 & \{f_1L_{jr} + \beta^2\omega L_jL_r + L^2\omega b_jb_r - L\beta\omega(L_jb_r + L_rb_j)\} D_i^r + \\
 & (L\beta\omega L_i - L^2\omega b_i)(r_{0j} + s_{0j}) - (L\beta\omega L_j - L^2\omega b_j)(r_{0i} + s_{0i}).
 \end{aligned} \tag{3.13}$$

Subtracting (3.12) from (3.10) and contracting the resulting equation with y^i , we obtain

$$\begin{aligned}
 & \{-f_1L_{jr} + L\beta\omega L_jb_r + L\beta\omega L_rb_j - \beta^2\omega L_jL_r - L^2\omega b_jb_r\} D^r - \\
 & \frac{1}{2}(L\beta\omega L_j - L^2\omega b_j)r_{00} + f_2r_{0j} = (f_1L_r + f_2b_r) D_j^r.
 \end{aligned} \tag{3.14}$$

Contracting (3.14) with y^j , we get

$$2(f_1L_r + f_2b_r) D^r = f_2r_{00}. \tag{3.15}$$

Adding (3.10) and (3.13) and contracting the resulting equation with y^j , we get

$$\{f_1L_{ir} + \beta^2\omega L_iL_r + L^2\omega b_ib_r - L\beta\omega(L_ib_r + L_rb_i)\} D^r = \frac{1}{2}(L^2\omega b_i - L\beta\omega L_i)r_{00} + f_2s_{i0}. \tag{3.16}$$

In view of $LL_{ir} = g_{ir} - L_iL_r$, the equation (3.16) can be written as

$$\begin{aligned}
 & \frac{f_1}{L} g_{ir} D^r + \left\{ \left(-\frac{f_1}{L} + \beta^2\omega \right) L_i - L\beta\omega b_i \right\} L_r D^r + \\
 & (L^2\omega b_i - L\beta\omega L_i) b_r D^r = \frac{1}{2}(L^2\omega b_i - L\beta\omega L_i) r_{00} + f_2s_{i0}.
 \end{aligned} \tag{3.17}$$

Contracting (3.17) with $b^i = g^{ij}b_j$, we get

$$\left(\frac{-f_1\beta}{L^2} - L\beta\omega\Delta \right) L_r D^r + \left(\frac{f_1}{L} + L^2\omega\Delta \right) b_r D^r = \frac{L^2\omega\Delta}{2} r_{00} + f_2s_0, \tag{3.18}$$

where $\Delta = b^2 - \frac{\beta^2}{L^2}$ and $s_0 = s_{r0}b^r$.

The equations (3.15) and (3.18) constitute the system of algebraic equations in $L_r D^r$ and $b_r D^r$ whose solution is given by

$$b_r D^r = \frac{(f_1f_2\beta + fL^3\omega\Delta)}{2f(f_1 + L^3\omega\Delta)} r_{00} + \frac{f_1f_2L^2}{f(f_1 + L^3\omega\Delta)} s_0 \tag{3.19}$$

and

$$L_r D^r = \frac{Lf_1f_2}{2f(f_1 + L^3\omega\Delta)} r_{00} - \frac{L^2f_2^2}{f(f_1 + L^3\omega\Delta)} s_0. \tag{3.20}$$

Contracting (3.17) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (3.19) and (3.20) respectively, we get

$$\begin{aligned}
 D^i = & \left\{ \frac{f_1f_2 - fL\beta\omega}{2f(f_1 + L^3\omega\Delta)} r_{00} - \frac{Lf_2(f_1f_2 - fL\beta\omega)}{ff_1(f_1 + L^3\omega\Delta)} s_0 \right\} y^i + \\
 & \left\{ \frac{L^3\omega}{2(f_1 + L^3\omega\Delta)} r_{00} - \frac{f_2L^4\omega}{f_1(f_1 + L^3\omega\Delta)} s_0 \right\} b^i + \frac{Lf_2}{f_1} s_0^i,
 \end{aligned} \tag{3.21}$$

where $l^i = \frac{y^i}{L}$.

Proposition 3.1. *The difference tensor $D^i = \overline{G}^i - G^i$ of any β -change of Finsler metric is given by (3.21).*

4 Projective Change of Finsler Metric

The Finsler space \bar{F}^n is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of \bar{F}^n . It is well known that the change $L \rightarrow \bar{L}$ is projective if $G^i = G^i + P(x, y)y^i$, where $P(x, y)$ is a homogeneous scalar function of degree one in y^i , called projective factor [3]. Thus, from (3.5) it follows that $L \rightarrow \bar{L}$ is projective iff $D^i = Py^i$.

Now, we consider that the β -change $L \rightarrow \bar{L} = f(L, \beta)$ is projective. Then, from equation (3.21), we have

$$Py^i = \left\{ \frac{f_1 f_2 - fL\beta\omega}{2f(f_1 + L^3\omega\Delta)} r_{00} - \frac{Lf_2(f_1 f_2 - fL\beta\omega)}{f f_1(f_1 + L^3\omega\Delta)} s_0 \right\} y^i + \left\{ \frac{L^3\omega}{2(f_1 + L^3\omega\Delta)} r_{00} - \frac{f_2 L^4\omega}{f_1(f_1 + L^3\omega\Delta)} s_0 \right\} b^i + \frac{Lf_2}{f_1} s_0^i, \quad (4.1)$$

Contracting (4.1) with $y_i (= g_{ij}y^j)$ and using the fact that $s_0^i y_i = 0$ and $y_i y^i = L^2$, we get

$$P = \frac{f_1 f_2}{2f(f_1 + L^3\omega\Delta)} r_{00} - \frac{f_2^2 L}{f(f_1 + L^3\omega\Delta)} s_0. \quad (4.2)$$

Putting the value of P from (4.2) in (4.1), we get

$$\beta\omega(f_1 r_{00} - 2f_2 L s_0) y^i = L^2\omega(f_1 r_{00} - 2f_2 L s_0) b^i + 2f_2(f_1 + L^3\omega\Delta) s_0^i. \quad (4.3)$$

Transvecting (4.3) by b_i , we get

$$r_{00} = \frac{-2f_2 s_0}{L^2\omega \Delta}, \quad \text{where} \quad \Delta = b^2 - \frac{\beta^2}{L^2} \neq 0. \quad (4.4)$$

Substituting the value of r_{00} from (4.4) in (4.2), we get

$$P = \frac{-f_2^2 s_0}{fL^2\omega \Delta}. \quad (4.5)$$

Eliminating P and r_{00} from (4.5), (4.4) and (4.1), we get

$$s_0^i = \left(b^i - \frac{\beta}{L^2} y^i \right) \frac{s_0}{\Delta}. \quad (4.6)$$

The equations (4.4) and (4.6) give the necessary conditions under which the β -change becomes a projective change. Conversely, if conditions (4.4) and (4.6) are satisfied, then putting these conditions in (3.21), we get

$$D^i = \frac{-f_2^2 s_0}{fL^3\omega \Delta} y^i, \quad \text{i.e.} \quad D^i = Py^i, \quad \text{where} \quad P = \frac{-f_2^2 s_0}{fL^3\omega \Delta}.$$

Thus \bar{F}^n is projective to F^n .

Theorem 4.1. *The β -change of Finsler space is projective if and only if (4.4) and (4.6) hold.*

Let us assume that L is the metric of a Riemannian space, that is, $L = \alpha = \sqrt{a_{ij}(x)y^i y^j}$. Then $\bar{L} = f(\alpha, \beta)$ which is the metric of any β -changed space. In this case $b_{i|j} = b_{i;j}$ where $;$ j denotes the covariant derivative with respect to Christoffel symbols constructed from Riemannian metric α . Thus r_{ij} and s_{ij} are functions of coordinates only, and in view of the Theorem 4.1, it follows that the Riemannian space is projective to Finsler space obtained from β -change (3.1) iff

$$r_{00} = \frac{-2f_2 s_0}{\alpha^2\omega \Delta} \quad \text{and} \quad s_0^i = -\left(\frac{\beta}{\alpha^2} y^i - b^i \right) \frac{s_0}{\Delta}, \quad \text{where} \quad \Delta = b^2 - \frac{\beta^2}{\alpha^2} \neq 0.$$

These equations may be written as

$$(a) \quad r_{00}(\beta^2 - b^2\alpha^2) = \frac{2f_2}{\omega}s_0, \quad (b) \quad s_0^i(\beta^2 - b^2\alpha^2) = (\beta y^i - \alpha^2 b^i)s_0. \quad (4.7)$$

The equation (4.7)(b) can be written as

$$\begin{aligned} & (s_j^i b_h b_k + s_h^i b_j b_k + s_k^i b_j b_h) - b^2(s_j^i a_{hk} + s_h^i a_{jk} + s_k^i a_{jh}) \\ &= \frac{1}{2}[(b_h s_k + b_k s_h)\delta_j^i + (b_j s_k + b_k s_j)\delta_h^i + \\ & \quad (b_j s_h + b_h s_j)\delta_k^i] - b^i(a_{hk}s_j + a_{hj}s_k + a_{jk}s_h). \end{aligned}$$

Contracting this equation with $i = j$, we get

$$(s_h b_k + s_k b_h) = 0, \quad \text{for } n > 2. \quad (4.8)$$

Transvection of (4.8) by b^h , we get $b^2 s_k = 0$, which implies that $s_k = 0$ provided $b^2 \neq 0$. Therefore, we have $s_0^i = 0$, $s_0 = 0$ and (4.7)(a) gives $r_{00} = 0$ as $\beta^2 - b^2\alpha^2 \neq 0$, consequently $r_{ij} = 0$, $s_{ij} = 0$. Hence $b_{i;j} = 0$, i.e. the pair (α, β) is parallel pair. Conversely, if $b_{i;j} = 0$ the equation (4.7)(a) and (b) hold identically. Thus, we have the following theorem

Theorem 4.2. *The Riemannian space with metric α is projective to the Finsler space with (α, β) -metric iff the (α, β) is a parallel pair.*

5 Particular Cases of One-form β

Let b_i be components of a parallel vector field in F^n i.e. $b_{i|j} = 0$. Therefore $r_{ij} = s_{ij} = 0$. Hence, the equation (3.21) gives $D^i = 0$ which implies that $\bar{G}^i = G^i$, $\bar{G}_j^i = G_j^i$ and $\bar{G}_{jk}^i = G_{jk}^i$. Thus, we have the following theorem

Theorem 5.1. *The Berwald connection $B\Gamma = (G_{jk}^i, G_j^i, 0)$ is invariant under β -change for parallel vector field b_i .*

Let b_i be a concurrent vector field in F^n [4, 9]. Then, we have (i) $b_{i|j} = -g_{ij}$ (ii) $b_i C_{jk}^i = 0$.

Thus, for a concurrent vector field $s_{ij} = 0$, $r_{ij} = -g_{ij}$ and therefore $r_{00} = -L^2$ and equation (3.21) reduces to

$$D^i = -\frac{(f_1 f_2 - f L \beta \omega) L^2}{2f(f_1 + L^3 \omega \Delta)} y^i - \frac{L^5 \omega}{2(f_1 + L^3 \omega \Delta)} b^i. \quad (5.1)$$

If $D^i = 0$, then equation (5.1) shows that b^i and y^i are linearly related. That is, there exists a scalar λ such that

$$b^i = \lambda y^i. \quad (5.2)$$

Since for a concurrent vector field the contravariant components b^i are also functions of x^i only [4, 9], differentiating (5.2) with respect to y^j , we get

$$(\dot{\partial}_j \lambda) y^i + \lambda \delta_j^i = 0.$$

Contracting it with respect to i and j and using the fact that λ is homogeneous function of degree -1 in y^i , we get

$$(n-1)\lambda = 0 \quad \text{i.e.} \quad \lambda = 0$$

which is not possible as in that case β vanishes. Hence, we have the following theorem

Theorem 5.2. *If b_i are components of a concurrent vector field, then the Berwald connection $B\Gamma$ is not invariant under β -change.*

Next, suppose that b_i is a gradient vector field so that $s_{ij} = \frac{1}{2}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}) = 0$. Then equation (3.21) reduces to

$$D^i = \frac{(f_1 f_2 - f L \beta \omega)}{2f(f_1 + L^3 \omega \Delta)} r_{00} y^i + \frac{L^3 \omega r_{00}}{2(f_1 + L^3 \omega \Delta)} b^i. \quad (5.3)$$

If $r_{00} \neq 0$ and $D^i = 0$, then we get the same result as given in theorem (5.2) for gradient vector field b_i but if r_{00} is also zero, then the Berwald connections $B\Gamma$ is invariant under β -change. Thus, we have the following theorem

Theorem 5.3. *If b_i are components of a gradient vector field, then the Berwald connection $B\Gamma$ is invariant if $r_{00} = 0$.*

6 Conclusion

Here, I introduced n -dimension Finsler space differentiable on a manifold along with a fundamental function. With some preliminaries and historical developments, we mainly focused on the β -change of Finsler metric and projective change of Finsler metric. The difference tensor of any β -change of Finsler metric is derived. The necessary and sufficient condition of the β -change of a Finsler space to be projective is presented. Some particular cases of β -change and Berwald connections are discussed.

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