An Existence Result for Henstock-Kurzweil-Stieltjes-$\diamond$-Double Integral of Interval-Valued Functions on Time Scales

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Abstract: *We employ the concept of interval-valued functions to state and prove an existence result for the Henstock-Kurzweil-Stieltjes-$\diamond$-double integral on time scales.*

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1 Introduction

The Henstock-Kurzweil integral is a generalization of Riemann integral that was studied independently by Henstock [5] and Kurzweil [6]. The Henstock-Kurzweil-Stieltjes integral is a generalized Riemann-Stieltjes integral which shares the same properties. The theory of interval analysis can be traced back to the celebrated book of Moore et al. [7]. In 2016, Yoon [11] presented some properties of interval-valued Henstock-Stieltjes integral on time scales. The Henstock-Kurzweil delta integral on time scales was introduced by Peterson and Thompson [9] and Henstock-Kurzweil integrals on time scales was studied by Thompson [10]. We relate the time scales version of integration to the usual form, most of the properties of a time scale integral can be realized by using the methods tailored to the time scale setting (see [2] [3], [4], [8], [9], [10]).

Some basic properties such as uniqueness and Bolzano Cauchy criterion of fuzzy Henstock-Kurzweil-Stieltjes-$\diamond$-double integral on time scales are stated and proved by the authors in [1].
In this paper, the authors are concerned with an existence result for Henstock-Kurzweil-Stieltjes-$\diamond$-double integral of interval-valued functions on time scales because of its various applications in the theory of integration.

A time scale $\mathbb{T}$ is any closed non-empty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and a rectangle $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, respectively. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 1.1. Let $\mathbb{T}_1, \mathbb{T}_2$ be two given time scales and let $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is an Hausdorff metric space with the metric (distance) d_H, define $d_H(A, B) = d_H((a, c), (b, d)) = ((b - a)^2 + (d - c)^2)$ as the distance between A and B. Then

$$\begin{aligned} d_H(A,B) &= \max\left(\left|A^- - B^-\right|, \left|A^+ - B^+\right|\right) \\ &= \max\left(\left|(a^-,c^-)-(b^-,d^-)\right|, \left|(a^+,c^+)-(b^+,d^+)\right|\right) \\ &= \max((b^- - a^-)^2 + (d^- - c^-)^2)^{\frac{1}{2}}, ((b^+ - a^+)^2 + (d^+ - c^+)^2)^{\frac{1}{2}}. \end{aligned}$$

We now introduce Henstock-Kurzweil-Stieltjes-◇-double integral over versions in $\mathbb{T}_1 \times \mathbb{T}_2$.

Definition 1.2. Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be a bounded function on $\mathcal{R}$ and let g be a non-decreasing function defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1,2,...,n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1,2,...,k$. Then

$$S(P_1,P_2,F,g_1,g_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as Henstock-Kurweil-Stieltjes-◇-double sum of F with respect to functions g_1 and g_2.
Let $P = P_1 \times P_2$, then the Henstock-Kurweil-Stieltjes-◇-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P,F,g_1,g_2)$.

Definition 1.3. Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} : t \in [a,b)_{\mathbb{T}_1}, s \in [c,d)_{\mathbb{T}_1}$. We say that F is Henstock-Kurzweil-Stieltjes-◇-double integrable with respect to non-decreasing functions g_1, g_2 defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ if there is a number α, a member of $\mathbb{R}$ such that for every $\varepsilon > 0$, there is a ◇-gauge δ (or γ) such that

$$d_H\left(S(P,F,g_1,g_2), I_0\right) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1,...,n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,...,k$ are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$.

We say that I_0 is the Henstock-Kurzweil-Stieltjes-◇-double integral of F with respect to g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t,s) \diamond g_1(t) \diamond g_2(s) = I_0.$$

2 The Main Results

We need the following definitions to prove an existence theorem of interval Henstock-Kurzweil-Stieltjes-◇-double integral on time scales.

Definition 2.1. A function $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is bounded on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 if there exists $M \geq 0$ in $I_{\mathbb{R}}$ such that

$$|F(t,s)| \leq M, \text{ for all } t,s \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}.$$

Definition 2.2. A function $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is continuous at $t_0, s_0 \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, if for any $\varepsilon > 0$ there exists a positive $\delta = \delta(t_0,s_0)$ such that whenever $t,s \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$,

$$d_H(F(t,s), F(t_0,s_0)) < \varepsilon$$

implies

$$d_H((t,s),(t_0,s_0)) < \delta.$$

If $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is uniformly continuous on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, then it is continuous on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$.

If P_1 and P_2 are tagged partitions of $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, then there exists $\mathcal{P}$ a collection of all divisions of $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. The variation of F over $\mathcal{P}$ is given by

$$\mathbf{var}(F,\mathcal{P}) = \sum_{P_1}\sum_{P_2} d_H(F(t,s),F(t_0,s_0)).$$

Note that for any division $\mathcal{P}$ of $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$, $\mathbf{var}(F,\mathcal{P})$ is a continuous function on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$.

Definition 2.3. A function $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is said to be of bounded variation on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ if

$$\mathbf{BV}_F = \mathbf{BV}(F,[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}) = \sup_{P\in\mathcal{P}([a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2})} \mathbf{var}(F,\mathcal{P})$$

is continuous on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$.

Theorem 2.1. *Let $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be of bounded variation on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. Then the variation of F is additive; that is, if $a \le \alpha \le b$ and $c \le \beta \le d$, then*

$$\mathbf{Var}(F;[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}) = \mathbf{Var}(F;[a,\alpha]_{\mathbb{T}_1} \times [c,\beta]_{\mathbb{T}_2} + \mathbf{Var}(F;[\alpha,b]_{\mathbb{T}_1} \times [\beta,d]_{\mathbb{T}_2}.$$

Proof. Suppose that $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is of bounded variation. Let $\alpha,\beta \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ and let P_1 and P_2 be two partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0,t_1,...,t_n\} \subset [a,b]_{\mathbb{T}_1}$ and $P_2 = \{s_0,s_1,...,s_n\} \subset [c,d]_{\mathbb{T}_2}$. Let $\{\xi_1,\xi_2,...,\xi_n\}$ denote an arbitrary selection of points from $[a,b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1},t_i)_{\mathbb{T}_1}, i = 1,2,...,n$. Similarly, let $\{\zeta_1,\zeta_2,...,\zeta_n\}$ denote an arbitrary selection of points from $[c,d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1},s_j)_{\mathbb{T}_2}, j = 1,2,...,k$. Then $P_1^{'} = \{t_0,t_1,\alpha...,t_n\} \subset [a,b]_{\mathbb{T}_1}$ and $P_2^{'} = \{s_0,s_1,\beta...,s_n\} \subset [c,d]_{\mathbb{T}_2}$ are refinements of P_1 and P_2 obtained by adjoining α and β to P_1 and P_2 respectively. Thus

$$\begin{aligned}\sum_{P}\sum_{P} d_H(F(t,s),F(t_0,s_0)) &\le \sum_{P_1}\sum_{P_2} d_H(F(t,s),F(t_0,s_0)) \\ &\quad + \sum_{P_1^{'}}\sum_{P_2^{'}} d_H(F(t,s),F(t_0,s_0))\end{aligned}$$

where P_1 and P_2 are partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ respectively and $P_1^{'}$ and $P_2^{'}$ are also partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ respectively. Then $P^{'} = P_1^{'} \cup P_2^{'}$ and that

$$\begin{aligned}\sum_{P_1}\sum_{P_2} d_H(F(t,s),F(t_0,s_0)) &\le \sup_{P\in\mathcal{P}(F;[a,\alpha]_{\mathbb{T}_1}\times[c,\beta]_{\mathbb{T}_2}} \left(\sum_{P}\sum_{P} d_H(F(t,s),F(t_0,s_0))\right) \\ &= \mathbf{var}(F;[\alpha,b]_{\mathbb{T}_1} \times [\beta,d]_{\mathbb{T}_2})\end{aligned}$$

and

$$\begin{aligned}\sum_{P_1^{'}}\sum_{P_2^{'}} d_H(F(t,s),F(t_0,s_0)) &\le \sup_{P\in\mathcal{P}(F;[a,\alpha]_{\mathbb{T}_1}\times[c,\beta]_{\mathbb{T}_2}} \left(\sum_{P}\sum_{P} d_H(F(t,s),F(t_0,s_0))\right) \\ &= \mathbf{var}(F;[a,\alpha]_{\mathbb{T}_1} \times [c,\beta]_{\mathbb{T}_2}).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{var}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}) &= \sup_{P\in\mathcal{P}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}}\left(\sum_P\sum_P d_H(F(t,s),F(t_0,s_0))\right)\\ &\leq \mathbf{var}(F;[\alpha,b]_{\mathbb{T}_1}\times[\beta,d]_{\mathbb{T}_2})+\mathbf{var}(F;[a,\alpha]_{\mathbb{T}_1}\times[c,\beta]_{\mathbb{T}_2}).\end{aligned}$$

On the other hand, for any $P_1^{'}=\{t_0,t_1,\alpha...,t_n\}\subset[a,b]_{\mathbb{T}_1}$ and $P_2^{'}=\{s_0,s_1,\beta...,s_n\}\subset[c,d]_{\mathbb{T}_2}$ are refinements of P_1 and P_2 obtained by adjoining α and β to P_1 and P_2 respectively. Then $P^{'}=P_1^{'}\cup P_2^{'}\in\mathcal{P}_{\alpha,\beta}([a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2})$ is the set of all divisions of $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ with α and β as the division points. Hence,

$$\begin{aligned}&\sup_{P'\in\mathcal{P}_{\alpha,\beta}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}}\left(\sum_{P'}\sum_{P'} d_H(F(t,s),F(t_0,s_0))\right)\\ \leq\ &\sup_{P\in\mathcal{P}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}}\left(\sum_P\sum_P d_H(F(t,s),F(t_0,s_0))\right)\\ =\ &\mathbf{var}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbf{var}(F;[\alpha,b]_{\mathbb{T}_1}\times[\beta,d]_{\mathbb{T}_2})+\mathbf{var}(F;[a,\alpha]_{\mathbb{T}_1}\times[c,\beta]_{\mathbb{T}_2})\\ \leq\ &\sup_{P'\in\mathcal{P}_{\alpha,\beta}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}}\left(\sum_{P'}\sum_{P'} d_H(F(t,s),F(t_0,s_0))\right)\\ \leq\ &\mathbf{var}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}).\end{aligned}$$

Therefore, combining the two inequalities, we have

$$\mathbf{var}(F;[\alpha,b]_{\mathbb{T}_1}\times[\beta,d]_{\mathbb{T}_2})+\mathbf{var}(F;[a,\alpha]_{\mathbb{T}_1}\times[c,\beta]_{\mathbb{T}_2})=\mathbf{var}(F;[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}).$$

This completes the proof. □

Theorem 2.2. *[Existence Theorem] Let $F:[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to I_{\mathbb{R}}$ be a continuous function and $g:[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to I_{\mathbb{R}}$ be of bounded variation on $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$, then F is Henstock-Kurzweil-Stieltjes-◇-double integrable on $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$.*

Proof. Let $\varepsilon>0$. Since g is of bounded variation, $\mathbf{Var}_g\in I_{\mathbb{R}}$ and $g=g_1\times g_2$. This means that there exists $M>0$ such that $\mathbf{Var}_g(t,s)\leq M$ for all $t,s\in[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$. Since F is continuous on $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$, for all $t_0,s_0\in[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ there exists a positive $\delta_0(t_0,s_0)$ such that whenever $t,s\in[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ with

$$d_H(t,s),(t_0,s_0))<\delta_0,$$

we have

$$d_H(F(t,s),F(t_0,s_0))<\varepsilon.$$

Let a positive gauge δ be defined on $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ by $\delta=\dfrac{\delta_0}{2}$, for all $t,s\in[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$. Let

$$P_1=\{([a,t_1],\xi_1),([t_1,t_2],\xi_2),...,([t_{n-1},b],\xi_n)\}\subset[a,b]_{\mathbb{T}_1}$$

and

$$P_2=\{([c,s_1],\zeta_1),([s_1,s_2],\zeta_2),...,([s_{k-1},d],\zeta_k)\}\subset[c,d]_{\mathbb{T}_2}$$

be δ-fine tagged divisions of $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$. Then, there exists a tagged division P_0 such that $P_1<P_0$ and $P_2<P_0$. Now, for every $([t_{i-1},t_i],\xi_i)\in d_H$ and $([s_{j-1},s_j],\zeta_j)\in d_H$; $i=1,2,...,n$; $j=1,2,...,k$. Now we have the difference

$$d_H((t_{i-1},s_{j-1}),(t_i,s_j)=F(\xi_i,\zeta_j)[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))]-S(F,g_i,P_{i,j})$$

where

$$P_{i,j} = \left\{ \left(\left[X_{q-1}^{(i,j)}, X_q^{(i,j)} \right], s_q^{(i,j)} \right) \quad , \quad X_0^{(i,j)} = (t_{i-1}, s_{j-1}), \ X_{m_i}^{(i,j)} = (t_i, s_j), \ q-1 < m_{i,j}$$

is a refinement of $([(t_{i-1}, s_{j-1}), (t_i, s_j)], (\xi_i, \zeta_j))$ in P_0. Then

$$d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) = \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} F(\xi_i, \zeta_j) - F(s_q^{(i,j)}) \right) \left(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}) \right).$$

Now, $s_q^{i,j}, (\xi_i, \zeta_j) \in ((t_{i-1}, s_{j-1}), (t_i, s_j)) \subseteq (\xi_i, \zeta_j) - \delta(\xi_i, \zeta_j), (\xi_i, \zeta_j) + \delta(\xi_i, \zeta_j)$ which implies that

$$\left| (\xi_i, \zeta_j) - s_q^{i,j} \right| \leq d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) < \delta(\xi_i, \zeta_j).$$

By continuity of F at (ξ_i, ζ_j),

$$\mid s_q^{i,j} - (\xi_i, \zeta_j) \mid < \delta(\xi_i, \zeta_j) = \frac{\delta_0(\xi_i, \zeta_j)}{2} < \delta_0(\xi_i, \zeta_j)$$

it implies that

$$d_H(F(s_q^{i,j}) - F(\xi_i, \zeta_j)) < \varepsilon.$$

So,

$$d_H((t_{i-1}, s_{j-1}), (t_i, s_j)) = \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\sum_{q=1}^{m_{i,j}} F(\xi_i, \zeta_j) - F(s_q^{(i,j)}) \right) \left(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}) \right).$$

Hence, by Theorem 2.1, we have

$$
\begin{aligned}
& d_H(S(P,F,g) - S(P_0,F,g)) \\
= \; & d_H\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))], \sum_{i=1}^{n}\sum_{j=1}^{k} S(P_{i,j},F,g)\right) \\
= \; & d_H\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\{F(\xi_i,\zeta_j)[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))], S(P_{i,j},F,g)\}\right) \\
= \; & \left|\sum_{i=1}^{n}\sum_{j=1}^{k} d_H((t_{i-1},s_{j-1}),(t_i,s_j))\right| \\
\leq \; & \sum_{i=1}^{n}\sum_{j=1}^{k} |d_H((t_{i-1},s_{j-1}),(t_i,s_j))| \\
= \; & \sum_{i=1}^{n}\sum_{j=1}^{k}\left|\sum_{q=1}^{m_{i,j}} F(\xi_i,\zeta_j) - F(s_q^{(i,j)})(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}))\right| \\
\leq \; & \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\sum_{q=1}^{m_{i,j}}\left|F(\xi_i,\zeta_j) - F(s_q^{(i,j)})(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}))\right|\right) \\
\leq \; & \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\sum_{q=1}^{m_{i,j}}\frac{\varepsilon}{K}\left|(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}))\right|\right) \\
\leq \; & \frac{\varepsilon}{K}\cdot\sum_{i=1}^{n}\sum_{j=1}^{k}\left(\sum_{q=1}^{m_{i,j}}\left|(g(X_q^{(i,j)} - g(X_{q-1}^{(i,j)}))\right|\right) \\
\leq \; & \frac{\varepsilon}{K}\cdot\sum_{i=1}^{n}\sum_{j=1}^{k} Var[g,(t_{i-1},s_{j-1}),(t_i,s_j)] \\
= \; & \frac{\varepsilon}{K}Var_g < \frac{\varepsilon}{K}K = \varepsilon.
\end{aligned}
$$

Similarly,

$$d_H(S(Q,F,g),S(P_0,F,g)) < \varepsilon.$$

Thus,

$$
\begin{aligned}
d_H(S(P,F,g),S(Q,F,g)) \; & \leq \; d_H(S(P,F,g),S(P_0,F,g)) + D(S(P_0,F,g),S(Q,F,g)) \\
& < \; \varepsilon + \varepsilon \\
& = \; 2\varepsilon.
\end{aligned}
$$

By Cauchy criterion theorem in [1], F is Henstock-Kurzweil-Stieltjes-◇-double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. □

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