Analytical Solution for Advection-Dispersion Equation of the Pollutant Concentration using Laplace Transformation

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Abstract: *We present simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration $C(x,t)$ in one dimension. In this model the water velocity in the x-direction is taken as a linear function of x and dispersion coefficient D as zero. In this paper by taking $k = 0$, k is the half saturated oxygen demand concentration for pollutant decay, we can apply the Laplace transformation and obtain the solution. The variation of $C(x,t)$ with different times t upto $t \to \infty$ (the steady state case) is taken into account advection-dispersion equation in our study.*

Keywords: Pollutant, Concentration, Laplace transformation, Dispersion, Analytical solution

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Nomenclature

A	Cross-section area of the river (m^2).
C	Pollutant concentrations ($kg.m^{-3}$).
D	Dispersion coefficient of pollutant in the x-direction ($m^2.day^{-1}$).
k	Half-saturated oxygen demand concentration for pollutant decay ($kg.m^{-3}$).
k_1	Degradation rate coefficient for pollutant (day^{-1}).
L	Polluted length of river (m).
s	Laplace transform variable.
p	Rate of pollution ($kg.m^{-3}$) at the origin .
q	Added pollutant rate along the river ($kg.m^{-1}$.day^{-1}).
t	Time (day).
u	Water velocity in the x-direction (m.day^{-1}).
x	Position (m).

1 Introduction

Pollution of rivers has become a matter of concern for scientists working in environmental engineering, hydrology, chemical engineering, geology, soil physics, and mathematics. If important hydraulic and chemical processes are examined together, analytical solutions of the mathematical models representing pollutant transport are hardly possible [21]. To forecast water quality and to give reliable tools for water quality management in affected areas, mathematical models have been used widely. The purpose of this study is to promote analytical solution of one-dimensional unsteady advection-dispersion equation using Laplace transforms method. It is a particular case of the research made by Pimpunchat et al. [11] which was carried at Tha Chin River in Thailand. The poor water quality of the Bagmati River in Kathmandu, Nepal was the motivation for this study. The rapid urban growth, continuous dumping of solid wastes, domestic sewage, industrial waste, insufficient waste-water treatment facilities, low levels of awareness are the main reasons for pollution in the Bagmati River (Fig. 1) [18].

Pimpunchat et al. [12] presented a mathematical model for river pollution comprising a coupled pair of nonlinear equations and had investigated the effect of aeration on the degradation of pollutant. In some simplified cases they had obtained analytic steady-state solutions. Carslaw and Jaeger [1] have derived

Figure 1: Polluted Bagmati River in Kathmandu, Nepal [18]

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analytical solutions for one-dimensional transport in composite media with Laplace transforms and with Green's functions. Marusic [10] presented a mathematical model for analyzing the hydrodynamics and pollutant dispersion in river-type systems. He was able to report changes in the pollutant concentration in the river with time. According to the study of Johari et al. [3], the results of one dimensional advection diffusion equation had successfully been used to predict the transportation of water pollution concentration by manipulating the velocity and diffusion parameters. Salkuyeh [16] put forward a method to find the exact solution of the system of ordinary differential equations obtained when we discretize the convection-diffusion equation with regard to the space variable. Pochai et al. [13] and Tabuenca et al. [19] presented the finite element method for solving the water pollution models in one and two dimensional water areas respectively. Van Genuchten and Alves [20] introduced analytical solutions for a physical system in a semi-infinite domain with zero initial concentration. Savovic and Djordjevich [17] presented numerical solution for the one dimensional advection-diffusion partial differential equation with variable coefficients in semi-infinite media. Kumar et al. [6] have derived analytical solutions for one-dimensional advection–diffusion equation in a longitudinal finite initially solute free domain with variable coefficients. Wadi et al. [21] presented simple analytical solutions for the unsteady advection-dispersion equations describing the pollutant concentration in one dimension. The solutions were obtained by using Laplace transformation technique. According to the study of Manitcharoen and Pimpunchat [9], the unsteady state solutions of pollutant concentration by considering advection-dispersion equations in one dimension were proposed by using the Laplace transform technique and the explicit finite difference technique, for analytical and numerical solutions, respectively. In recent years many techniques have been developed to find the solution of partial differential equations [4, 5, 14, 15]. One of such techniques is Laplace transform, which is a very useful technique.

2 Mathematical Model

The water pollution or the concentration of the pollutant $C(x,t)$ is assumed to vary with time t (days) along the length of the river $L(m)$ (the polluted length of the river) where it is assumed that the rate of pollutant addition along the river $q(kg/m$ day) is constant. An unsteady flow of water pollutant concentration in one dimension can be described by advection-dispersion equation [12]:

$$\frac{\partial(AC)}{\partial t} = D\frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(uAC)}{\partial x} - k_1\frac{X}{X+k}AC + qH(x); \quad 0 \le x < L, t > 0 \tag{1}$$

where $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{if} \quad 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Here, u is the water velocity in x- direction, C is the concentration of pollutant, D is the dispersion coefficient of pollutant in x- direction, k_1 is the degradation rate coefficient of pollutant, q is the added pollutant rate along the river, k is the half saturated oxygen demand concentration for pollutant decay, X is the concentration of the dissolved oxygen within the river and A is the cross-section of area of river. We assume the stream reach is considered to be homogeneous system. So we take the parameters A, q, D, k_1 as constants over time and space [8].

In the general case when $k \neq 0$, it will be impossible to use Laplace transform to suggest an exact solution [21]. We apply Laplace transformation by taking $k = 0$. In this model we take zero dispersion D (i.e. $D = 0$) [11]. Using these above conditions, the model equation (1) becomes:

$$\frac{\partial(AC)}{\partial t} = -\frac{\partial(uAC)}{\partial x} - k_1 AC + q; \quad 0 \leq x < L, t > 0 \tag{3}$$

Let us consider $u(x,t) = 1 + ax$ [2] for water velocity, where a is non-zero real constant has the dimension of inverse of space variable. Equation (3) can be written as

$$\frac{\partial(C)}{\partial t} = -u\frac{\partial(C)}{\partial x} - Ca - k_1 C + \frac{q}{A}; \quad 0 \leq x < L, t > 0 \tag{4}$$

This equation is solved under the initial and boundary conditions as:

$$C(x,0) = 0; \quad x \geq 0 \tag{5}$$

$$C(0,t) = p; \quad t > 0 \tag{6}$$

where $C(x,t)$ is the pollutant concentration for the case when dispersion coefficient $D = 0$, the initial rate of pollution along the river is supposed zero and p is the rate of pollution at the origin.

2.1 The analytical solution

Laplace transformation technique is defined by equation (7), and is used to get the analytical solution. The Laplace transformation may be defined as: If $f(x,t)$ is any function defined in $a \leq x \leq b$ and $t > 0$, then its Laplace transform with respect to t is denoted by:

$$L\{f(x,t)\} = F(x,s) = \int_0^\infty e^{-st} f(x,t)dt, s \succ 0 \tag{7}$$

where s is called the transform variable [7]. The inverse Laplace transformation is denoted by $L^{-1}\{F(x,s)\} = f(x,t)$ and defined by the complex variable:

$$L^{-1}\{F(x,s)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(x,s)e^{-st} f(x,t)ds, c \succ 0 \tag{8}$$

Applying Laplace transformation to equation (4) gives:

$$s\tilde{C}(x,s) - C(x,0) = -u\frac{\partial \tilde{C}}{\partial x} - \tilde{C}(x,s)a - k_1\tilde{C}(x,s) + \frac{q}{As}$$

$$s\tilde{C}(x,s) - C(x,0) = -u\frac{\partial \tilde{C}}{\partial x} - (a + k_1)\tilde{C}(x,s) + \frac{q}{As}$$

Using (5), we get,

$$s\tilde{C}(x,s) = -u\frac{\partial \tilde{C}}{\partial x} - (a+k_1)\tilde{C}(x,s) + \frac{q}{As}$$

$$\frac{\partial \tilde{C}}{\partial x} + \left(\frac{s+a+k_1}{1+ax}\right)\tilde{C} = \frac{q}{As(1+ax)}$$

Integrating factor $(I.F.) = e^{\int (\frac{s+a+k_1}{1+ax})dx} = (1+ax)^{(\frac{s+a+k_1}{a})}$
Thus

$$\tilde{C}(1+ax)^{(\frac{s+a+k_1}{a})} = \frac{q}{As}\int (1+ax)^{(\frac{s+k_1}{a})}dx$$

$$\tilde{C}(1+ax)^{(\frac{s+a+k_1}{a})} = \frac{aq}{As(s+a+k_1)}(1+ax)^{(\frac{s+a+k_1}{a})} + C_1$$

where C_1 is the constant of integration.
Therefore

$$\tilde{C}(x,s) = \frac{aq}{As(s+a+k_1)} + C_1(1+ax)^{-(\frac{s+a+k_1}{a})} \tag{9}$$

Taking laplace transform of (6)

$$\tilde{C}(0,s) = \frac{p}{s} \tag{10}$$

Using (10) on (9), we get,

$$\frac{p}{s} = \frac{aq}{As(s+a+k_1)} + C_1$$

Therefore

$$C_1 = \frac{p}{s} - \frac{aq}{As(s+a+k_1)}$$

Thus

$$\tilde{C}(x,s) = \frac{aq}{As(s+a+k_1)} + \frac{p}{s}(1+ax)^{-(\frac{s+a+k_1}{a})} - \frac{aq}{As(s+a+k_1)}(1+ax)^{-(\frac{s+a+k_1}{a})} \tag{11}$$

The inverse of Laplace transform is

$$C(x,t) = \frac{aq}{A}\left[\frac{1}{k_1+a} - \frac{1}{k_1+a}e^{-(k_1+a)t}\right] + p(1+ax)^{-(\frac{k_1+a}{a})}H\left(t+\frac{log(1+ax)}{a}\right) - \frac{aq}{A}\left[\left(\frac{1}{k_1+a} - \frac{1}{k_1+a}e^{-(k_1+a)t}\right)(1+ax)^{-(\frac{k_1+a}{a})}H\left(t+\frac{log(1+ax)}{a}\right)\right]$$

where

$$H\left(t+\frac{log(1+ax)}{a}\right)$$

is Heaviside function. This is used to capture the fact that pollutant is discharged for $\left(t+\frac{log(1+ax)}{a}\right) > 0$ only.

$$C(x,t) = \frac{aq}{A(k_1+a)} - \frac{aq}{A(k_1+a)}e^{-(k_1+a)t}$$

$$+p(1+ax)^{-(\frac{k_1+a}{a})} - \frac{aq}{A(k_1+a)}(1+ax)^{-(\frac{k_1+a}{a})} + \frac{aq}{A(k_1+a)}e^{-(k_1+a)t}(1+ax)^{-(\frac{k_1+a}{a})} \tag{12}$$

which can be written as

$$C(x^*,t) = q^* - q^*e^{-k_1^*t} + px^* - q^*x^* + q^*e^{-k_1^*t}x^*$$

where

$$k_1^* = (k_1 + a), \quad q^* = \frac{aq}{Ak_1^*}, \quad x^* = (1 + ax)^{-\frac{k_1^*}{a}}$$

$$\therefore C(x^*, t) = px^* + q^*(1 - x^*)(1 - e^{-k_1^* t}) \tag{13}$$

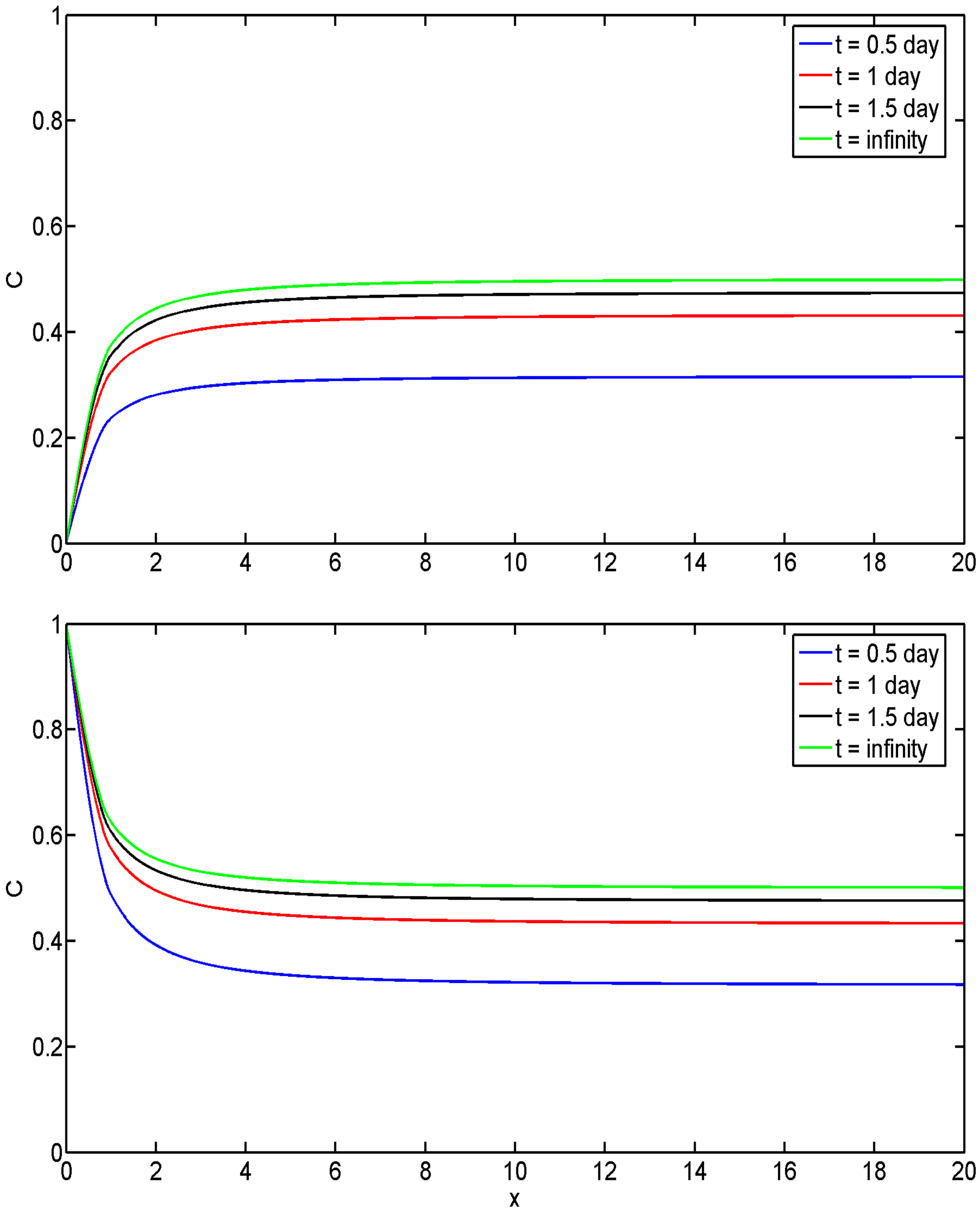

Figure 2: Analytical solution with dispersion for C at different time described by the equation (12) **top:** p=0, **bottom:** p=1.

2.1.1 Steady state case

Equation (12) for the steady state when $t \to \infty$ gives:

$$C(x) = \frac{aq}{A(k_1 + a)} + p(1 + ax)^{-\left(\frac{k_1+a}{a}\right)} - \frac{aq}{A(k_1 + a)}(1 + ax)^{-\left(\frac{k_1+a}{a}\right)} \tag{14}$$

For the special case when $p = 0$ the equation (14) will be the same as that obtained by Wadi et al. [21].

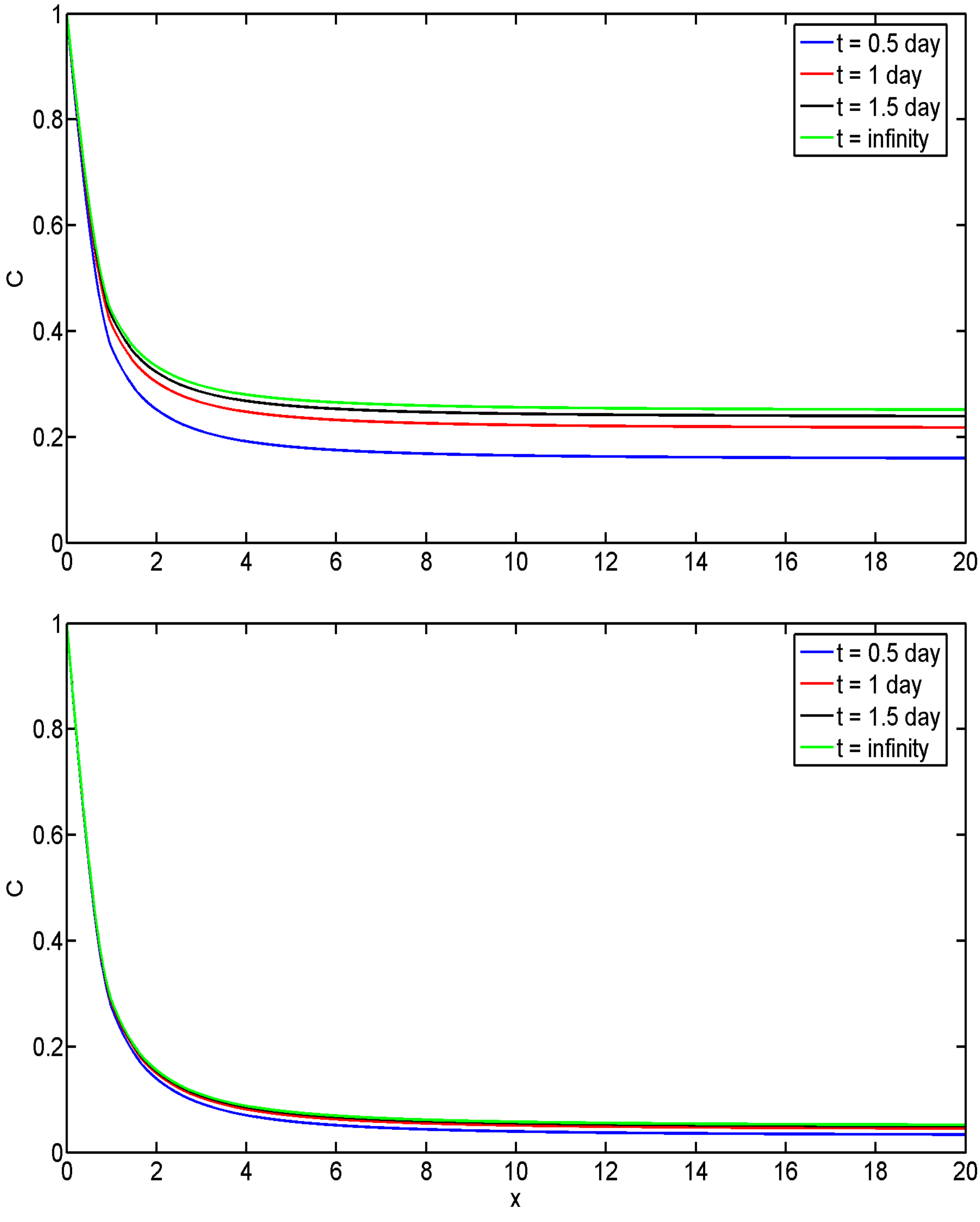

Figure 3: Analytical solution with dispersion for C at different time described by the equation (12) **top:** q=0.5, **bottom:** q=0.1.

3 Result and Discussion

We present simple analytical solution for the unsteady advection dispersion equation describing the pollutant concentration $C(x,t)$ in one dimension using Laplace transformation method for which zero dispersion D is taken. From solution we obtained variation of C with time and space. The time is given in days and the values of the concentration of pollutant are measured in $(kg.m^{-3})$. In general, the pollutant concentration is given by equation (12).

Figure 2 on the top shows the variation of C in the range $0 \leq x \leq 20$ with the time t described by equation (12) for $t = 0.5$, 1.0, 1.5 (days) and $t \to \infty$. To test our model we set the parameters A, q, a, k to be 1 and $p = 0$ [11]. From figure 2 on the top, for $x \geq 0$ as t increases C increases and reaches its maximum value as $t \to \infty$. In general, the concentration of pollutant increases as x increases. From figure 2 on the bottom, we see that: as x increases the value of C decreases for any time t, it reaches a constant value near the sink. The effect of the time t is very small near the upstream and dominant near the downstream. As t increases the value of C increases at any cross-section of the river. This result is in good agreement with that reported by Wadi et al. [21].

Figure 3 on the top shows the variation of C along the river from source up to sink at different times $t = 0.5$, 1.0, 1.5 (days) and $t \to \infty$ and added pollutant rate along the river $q = 0.5$ and figure 3 on the bottom shows the variation of C along the river with $q = 0.1$. From figure 3, if the added pollutant rate along the river q decreases, the variations of C at different times come to near at each other. If q is in very small amount the variation of C along the river at different times coincide to each other.

4 Conclusion

Simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration $C(x,t)$ in one dimension is derived by using Laplace transformation technique. We have obtained analytic unsteady solution by taking the water velocity u in the x-direction as a linear function of x and dispersion coefficient D as zero. Numerical studies show that the variation of C with time t is caused only by pure convection and rate of pollutant addition along the river and if the added pollutant rate along the river q is in very small amount, the variation of C along the river at different times coincide to each other.

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