Establishing Infinite Methods of Construction of Magic Squares

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Abstract: *Here we have established infinite methods of building doubly even magic squares from doubly even magic squares of n order ($n \geqslant 20$) which are formed by blocks of order four whose sums of elements of lines, columns and diagonals are all equal at $2n^2+2$. Such a characteristic of these special magic squares causes a large production of other magic squares.*

Keywords: Arithmetic progressions, Doubly even magic squares, Dual of Lohans' magic squares, Parity

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1 Introduction

Here we present new general methods which builds, for each order, new types of magic squares hitherto unknown. A *magic square of order n* (or *normal magic square*) is a square matrix formed by the numbers $1, 2, 3, ..., n^2$ and such that the sum of the numbers of each row, each column and each of the two diagonals is equal to $c_n = \dfrac{n^3+n}{2}$. We call c_n of *magic constant.* The magic square is *non-normal* when the sum of the numbers in lines, columns and diagonals are all the same, however, not equal to $c_n = \dfrac{n^3+n}{2}$ or the set of numbers that form it is not $I_n = \{1, 2, 3, ..., n\}$. We call the aforementioned sums of *totals.* If $n = 4k$, k positive natural number, the magic square is of type *doubly even magic square.*

2 Main results

Definition 1 (Lohans' magic squares of n order (MIRANDA, 2020a)). Let $n = 4k, k \in \mathbb{N}^*$ and $i, j \in \left\{4t, t \in I_{\frac{n}{4}}\right\}$. Consider the square matrix

$$M_n = (m_{u,v})_{u,v \in I_n} = \begin{pmatrix} M_{4,4} & \dots & M_{4,n} \\ \vdots & \ddots & \vdots \\ M_{n,4} & \cdots & M_{n,n} \end{pmatrix} \tag{1}$$

of order n determined by blocks of order 4 given by

$$M_{i,j} = \begin{pmatrix} n^2-2in+2n-2j+2 & 2in-n-2j+1 & n^2-2in+2n-2j & 2in-n-2j-1 \\ 2in-n+2j-1 & n^2-2in+2j & 2in-n+2j+1 & n^2-2in+2j+2 \\ n^2-2in-2j+2 & 2in+n-2j+1 & n^2-2in-2j & 2in+n-2j-1 \\ 2in+n+2j-1 & n^2-2in-2n+2j & 2in+n+2j+1 & n^2-2in-2n+2j+2 \end{pmatrix} \tag{2}$$

Then M_n above is a magic square called *Lohans' magic square of n order.*

The demonstration that M_n is a magic square can be found in (MIRANDA, 2020a) shared with (MIRANDA, 2020b).

Proposition 1 (Dual of Lohans' magic square of n order (MIRANDA, 2020b)). Let $n = 4k, k \in \mathbb{N}^*$ and $i, j \in \left\{4t, t \in I_{\frac{n}{4}}\right\}$. Consider the square matrix

$$D_n = (d_{u,v})_{u,v \in I_n} = \begin{pmatrix} D_{4,4} & ... & D_{4,n} \\ \vdots & \ddots & \vdots \\ D_{n,4} & \cdots & D_{n,n} \end{pmatrix} \tag{3}$$

of order n determined by blocks of order 4 given by

$$D_{i,j} = \begin{pmatrix} (i-3)n-(j-4) & (i-3)n+(j-2) & (n-i+2)n+(j-1) & (n+4-j)n-(j-1) \\ (n+4-i)n-(j-3) & (n-i+2)n+(j-3) & (i-3)n+j & (i-3)n-(j-2) \\ (i-1)n+(j-2) & (i-1)n-(j-4) & (n+2-i)n-(j-1) & (n-i)n+(j-1) \\ (n-i)n+(j-3) & (n+2-i)n-(j-3) & (i-1)n-(j-2) & (i-1)n+j \end{pmatrix} \tag{4}$$

Then: i) If $n > 4$, $D_{i,j}$ is a non-normal magic square with a total equal to $2n^2 + 2$ for each pair $i, j \in \left\{4t, t \in I_{\frac{n}{4}}\right\}$. If $n = 4$, $D_{i,j}$ is magic square; ii) D_n above is a magic square.

Proof. i) We have $(i-3)n-(j-4)+(i-3)n+(j-2)+(n-i+2)n+(j-1)+(n+4-j)n-(j-1) = 2n^2+2$. It is the same with the other three lines. We also have $(i-3)n-(j-4)+(n+4-i)n-(j-3)+(i-1)n+(j-2)+(n-i)n+(j-3) = 2n^2+2$. It is the same with the other three columns. Similarly $(i-3)n-(j-4)+(n-i+2)n+(j-3)+(n+2-i)n-(j-1)+(i-1)n+j = 2n^2+2$ and $(n-i)n+(j-3)+(i-1)n-(j-4)+(i-3)n+j+(n+4-j)n-(j-1) = 2n^2+2$. Therefore, $D_{i,j}$ is a non-normal magic square if $n \neq 4$ and normal if $n = 4$.

ii) The sum of the numbers of any line of D_n is equal to the sum of the numbers of $n/4$ matrix lines of type $D_{i,j}$ with j ranging from 4 to n traversing multiples of 4. Now, these $n/4$ sums are all equal to $2n^2+2$. Therefore, the sum of the numbers of any line of D_n is equal to $(n/4)(2n^2+2) = (n^3+n)/2 = c_n$, the magic constant. Analogous result is checked for any column. In the case of diagonals, the sum of the elements of the main diagonal of D_n is equal to the sum of the sums of the main diagonals of $D_{4,4}$, $D_{8,8}$, $D_{12,12}$,...,$D_{n,n}$. However, each of these has sum of diagonal numbers equal to $2n^2+2$. Therefore, the sum of the elements of the main diagonal of D_n is $(n/4)(2n^2+2) = c_n$. Similarly, we can prove that the sum of the elements of the secondary diagonal of D_n is also equal to the magic constant c_n.

Remark 1. The fact that the sum of the elements of all rows, columns and diagonals of all sub-squares of order four $D_{i,j}$ are equal to $2n^2+2$ makes the magic square D_n generate other magic squares of order n. In (MIRANDA, 2020c) an exhibition is presented in this sense. Below, we present some derived methods.

Definition 2. Let D be the dual of Lohans magic square of n order, as set out in (4) above. For the simultaneous exchange of: a) $(i-3)n-(j-4)$ with $(i-3)n+(j-2)$ and $(i-1)n+(j-2)$ with $(i-1)n-(j-4)$ we call procedure PE; b) $(i-3)n+j$ with $(i-3)n-(j-2)$ and $(i-1)n-(j-2)$ with $(i-1)n+j$ we call procedure PD; c) The union of the four exchanges mentioned in the previous items is called the PT procedure.

Remark 2. Note that with PE the main diagonal of $D_{i,j}$ wins $2j-6$ units, with PD loses $2j-2$ and with PT loses 4 units. The same occurs with the secondary diagonal of $D_{i,j}$.

Proposition 2. Let D_n be the dual of Lohans' magic square of n order ($n \geqslant 20$). Let's make in $D_{1,1}$ and $D_{2,2}$ the procedure PE and, in $D_{3,3}$, $D_{4,4}$ and $D_{5,5}$, the PT procedure. Then the matrix resulting from this set of procedures is a magic square.
Proof. First of all, note that none of the procedures changes the sum of the elements of the columns or rows of any of the five sub-squares $D_{i,j}$ involved. On the other hand, with the two PE procedures the main diagonal gains 12 units and, with the three PT procedures, loses 12 units. Therefore, the main diagonal does not change the sum of all its elements. Therefore, the matrix that results from the five procedures is a magic square.

Remark 3. Instead of $D_{3,3}$, $D_{4,4}$ and $D_{5,5}$, we could have chosen other possibilities to do the PT procedure three times, being able to build $(-2+n/4)(-3+n/4)(-4+n/4)$ magic squares. The number of methods for constructing magic squares is therefore greater than any fixed natural number. Instead of infinite methods, we can also speak of infinite variants of the method.

Proposition 3. Let D_n be the dual of Lohans' magic square of n order ($n \geqslant 20$). Let's make in $D_{1,1}$ and $D_{3,3}$ the procedure PD and in $D_{2,2}$ and $D_{3,3}$ the procedure PE . With both procedures PD the main diagonal loses 28 units and PE gains 28, with no change in the sum of the elements of the main diagonal. With $n > 20$ the secondary diagonal of D_n does not change. We could also have chosen $D_{1+k,1+k}$ and $D_{3+k,3+k}$ instead of $D_{1,1}$ e $D_{3,3}$ e , $D_{2+k,2+k}$ and $D_{3+k,3+k}$ instead of $D_{2,2}$ and in $D_{3,3}$ $(0 \leqslant k \leqslant -3 + \frac{n}{4})$, as long as the secondary diagonal does not change the sum of its elements. We could also have chosen $D_{1,1}$ and $D_{2,2}$ to apply the procedure PD and, $D_{1,1}$ and $D_{3,3}$, to apply the procedure PE, provided that the sum of the elements of the secondary diagonal does not change either. We could also opt for translations. In any case, once the four procedures mentioned above are performed, magic squares always result in a total of $-4 + n/2$ magic squares, as long as the sum of the secondary diagonal elements does not change. It should be noted that when n is a multiple of 8 the sum of the elements of the secondary diagonal never changes.

Remark 4. Many combinations are possible in the two propositions above, generating many other additional magic squares. Next, we establish an equation that generates all possible magic squares through these three procedures. Let D_n be the dual of Lohans' magic square of n order ($n \geqslant 20$). Let x be the number of procedures PE, y the number of procedures PD and z the number of procedures PT. So that the main diagonal does not change the sum of its elements, we must have $\sum_{k=1}^{x} (2j_k - 6) - \sum_{k=1}^{y} (2s_k - 2) - 4z = 0$. This equation is equivalent to

$$\sum_{k=1}^{x} j_k - \sum_{k=1}^{y} s_k = 3x - y + 2z; x, y, z \in I_{n/4}; j_k, s_k \in \left\{4u : u \in I_{n/4}\right\}. \qquad (5)$$

If no procedures are performed that alter the sum of the elements of the secondary diagonal or repetitions of procedures, the solutions obtained from (5) will generate magic squares. One way to avoid changing in the sum of the elements of the secondary diagonal is to avoid any of the three procedures in $D_{(n+4)/8,(n+4)/8}$.

3 Discussion

The dual Lohans' magic squares have the remarkable property established in Proposition 1. This property makes the doubly even magic squares to be demonstrably abundant among the three main types of magic squares. However, we do not know whether this property extends to other orders. We also do not know how the number of magic squares obtained by equation (5) grows. It should be noted that we do not obtain infinite methods for any fixed n order. What we get are several sequences of methods which are valid only after a certain value of n.

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