Family of Optimal Fourth Order Methods for Multiple Roots and their Dynamics

Prem Bahadur Chand[1,2], Kriti Sethi [3]

[1] National Academy of Science and Technology, Dhangadhi, Kailali, Nepal
[2] Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi, India
[3] Department of Mathematics, Guru Tegh Bahadur Institute of Technology, New Delhi, India, (Email: kritisethi90@gmail.com)

Correspondence to: Prem Bahadur Chand, Email: premchand13@gmail.com

Abstract: *In this paper, we propose a family of fourth order method for solving non-linear equations with multiple roots. The method is based on the arithmetic mean of Weerakoon method and Chebyshev method for multiple roots. Some numerical examples are provided in support of the theoretical results. The numerical results obtained by the method for different values of the parameter are compared with some known methods. The dynamical behaviour of methods is discussed and basins of attraction around the multiple roots for some polynomial is shown at the end of the work.*

Keywords: Non-linear equations, multiple roots, Weerakoon method, Chebyshev method.

1 Introduction

The following Newton's method is the well known iterative method for solving non-linear equation $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

If f has simple zero then the above method converges quadratically. There are numerous methods available with higher order each one claims to be the better than the other in some or the other aspect. We mention here two cubically convergent methods, namely Chebyshev's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 \cdot f''(x_n)}{2f'(x_n)^3}. \tag{2}$$

and the method of Weerakoon and Fernand[11] given by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{aligned} \tag{3}$$

It is known that all the three methods described above are only linearly convergent in case the multiplicity of roots is more than one. For a known multiplicity m, Liu et al. [6], presented a fourth order method given by:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ w_n &= \left(\frac{f'(y_n)}{f'(x_n)}\right)^{\frac{1}{m-1}}, \\ G &= w_n + \frac{2m}{m-1} \cdot w_n^2, \\ x_{n+1} &= y_n - m\Big(G \cdot \frac{f(x_n)}{f'(x_n)}\Big). \end{aligned} \tag{4}$$

In 2010, Sharma et al. [8] presented a fourth order method based on modified Jarrat's method [5] given by:

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - a_1 w_1(x_n) - a_2 w_2(x_n) - a_3\frac{w_2^2(x_n)}{w_1(x_n)}, \end{aligned} \tag{5}$$

where

$$w_1(x_n) = \frac{f(x_n)}{f'(x_n)}, \quad and \quad w_2(x_n) = \frac{f(x_n)}{f'(y_n)}$$

and the values of the parameters are given by

$$\begin{aligned} a_1 &= \frac{1}{8}m\left(m^3 - 4m + 8\right), \\ a_2 &= \frac{1}{4}(-m)(m-1)(m+2)^2\left(\frac{m}{m+2}\right)^m, \\ a_3 &= \frac{1}{8}m(m+2)^3\left(\frac{m}{m+2}\right)^{2m}. \end{aligned}$$

Similarly, Li et al. [7] also presented the following fourth order method:

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - a_1\cdot\frac{f(x_n)}{f'(y_n)} - \frac{f(x_n)}{a_2\cdot f'(x_n) + a_3\cdot f'(y_n)}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_1 &= -\frac{\left(m\left(m^4 + 4m^3 - 16m - 16\right)\right)\left(\frac{m}{m+2}\right)^m}{2\left(m^3 - 4m + 8\right)}, \\ a_2 &= -\frac{\left(m^3 - 4m + 8\right)^2}{m\left(m^4 + 4m^3 - 4m^2 - 16m + 16\right)\left(m^2 + 2m - 4\right)}, \\ a_3 &= \frac{m^2\left(m^3 - 4m + 8\right)\left(\frac{m+2}{m}\right)^m}{\left(m^4 + 4m^3 - 4m^2 - 16m + 16\right)\left(m^2 + 2m - 4\right)}. \end{aligned}$$

In the present paper, we propose a new optimal fourth order iterative method for finding multiple roots of $f(x) = 0$. Our method is based upon (2) and (3). The new aspect of our method that we discuss is its dynamical behavior, which was not done for the methods (4), (5) and (6).

2 Development of Methods and Their Convergence Analysis

To develop our method, we proceed as follows. We first rewrite method (3) by involving the multiplicity $m \geq 1$ as

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{aligned} \tag{7}$$

Again, the method (2) for multiple roots, as proposed by Behl et. al. [2], is given by

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)\big((5m+2)f'(x_n) - (m+2)f'(y_n)\big)}{4mf'(x_n)^2}. \end{aligned} \tag{8}$$

Now, taking the arithmetic mean of (7) and (8), we obtain

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{1}{2}\left[\frac{2f(x_n)}{f'(x_n)+f'(y_n)} + \frac{f(x_n)\big((5m+2)f'(x_n) - (m+2)f'(y_n)\big)}{4mf'(x_n)^2}\right]. \end{aligned} \tag{9}$$

It can be worked out that the error equation for the method (9) is given by

$$e_{n+1} = \frac{\Big(m^m(m+2)^{2-m} + \frac{8m^{m+2}(m+2)}{(m+2)m^m+(m+2)^m m} + m(m(8m-13)-2)\Big)}{8m^3} e_n + O(e_n^2)$$

so that the method (9) is linearly convergent for multiple zeros. We introduce some parameters to increase its order of convergence. Precisely, we propose the following method:

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{1}{2}\left[\frac{2a_1 f(x_n)}{a_2 f'(x_n)+f'(y_n)} + \frac{f(x_n)\big(a_3(5m+2)f'(x_n) - (m+2)a_4 f'(y_n)\big)}{4mf'(x_n)^2}\right]. \end{aligned} \tag{10}$$

We prove the following:

Theorem 1. *Let the function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for the open interval D has a zero α with multiplicity $m \geq 1$. Let $f(x)$ has sufficient number of smooth derivatives in the interval D and the initial point x_0 is close enough to α. Then the order of convergence of the method* (10) *is four for the following values of the parameters:*

$$\begin{aligned} a_1 &= -\frac{1}{8}m^{m-11}(m+2)^{-4m}\left(a_4 m^m(m+2) - 2m^4(m+2)^m\right)^3, \\ a_2 &= m^{m-5}(m+2)^{-2m}\left(-a_4(m+2)m^m - (m+2)^m m^5\right), \\ a_3 &= \frac{-a_4^2 m^{2m}(m+2)^{2-2m} + a_4 m^{m+4}(m+6)(m+2)^{1-m} - 4(m-2)m^7}{m^5(5m+2)}. \end{aligned}$$

Proof. Let, $f(x) = 0$ has a multiple root α of multiplicity m, then $f^{(i)}(\alpha) = 0$ for $i = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(\alpha) \neq 0$. Write $e_n = x_n - \alpha$ and $d_n = y_n - \alpha$, and $c_j = \dfrac{m!}{(m+j)!}\dfrac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$. Using Taylor series expansion of $f(x_n)$ about α, we obtain

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^m\left(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4\right) + O(e_n^5) \tag{11}$$

and

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{m!}e_n^{m-1}\left(m + (m+1)c_1 e_n + (m+2)c_2 e_n^2 + (m+3)c_3 e_n^3 + (m+4)c_4 e_n^4\right) + O(e_n^5). \tag{12}$$

Using first equation of (10), we obtain

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{m!}d_n^{m-1}\left(m + (m+1)c_1 d_n + (m+2)c_2 d_n^2 + (m+3)c_3 d_n^3 + (m+4)c_4 d_n^4\right) + O(d_n^5). \tag{13}$$

Now using (11), (12) and (13) in (10), we obtain the error equation of the method (10) as

$$e_{n+1} = A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O(e_n^5), \tag{14}$$

where

$$\begin{aligned}
A_1 &= \frac{1}{2}\Big[-\frac{2a_1}{a_2 m + (m+2)\lambda} + \frac{a_4\lambda(m+2)^2 - a_3 m(5m+2)}{4m^3} + 2\Big],\\
A_2 &= \frac{c_1}{8m^5}\Big[\frac{8a_1 m^3\left(a_2 m^2 + \left(m^2+2m-4\right)\lambda\right) - a_4\lambda\left(m^3+4m^2+8m+8\right)\left(a_2 m + (m+2)\lambda\right)^2}{\left(a_2 m + (m+2)\lambda\right)^2}\\
&\quad + a_3(5m+2)m^2\Big],\\
A_3 &= \frac{1}{2}\Big[\frac{2a_1 c_1^2\left(a_2 m^2(m+1) + \left(m^3+3m^2+2m-4\right)\lambda\right)}{m^2\left(a_2 m + (m+2)\lambda\right)^2} - \frac{2a_1 c_2}{a_2 m + (m+2)\lambda}\\
&\quad + \frac{c_2}{2m^5}\Big(a_3 m^2(5m+2) - a_4\lambda\left(m^3+4m^2+8m+8\right)\Big)\\
&\quad + \frac{c_1^2}{4m^7}\Big(a_4\lambda\left(m^5+5m^4+12m^3+16m^2+16m+16\right) - a_3 m^3\left(5m^2+7m+2\right)\Big)\\
&\quad - \frac{2a_1}{m^4\left(a_2 m + (m+2)\lambda\right)^3}\Big(c_1^2\big(a_2 m^2(m+1) + \left(m^3+3m^2+2m-4\right)\lambda\big)^2\\
&\quad - \left(a_2 m + (m+2)\lambda\right)\left(c_2 m^2\left(a_2 m^2(m+2) + \left(m^3+4m^2+4m-8\right)\lambda\right) - 4c_1^2(m-2)\lambda\right)\Big)\Big]
\end{aligned}$$

and

$$\begin{aligned}
A_4 = \frac{1}{2}\Bigg[&\frac{2a_1 c_1 c_2\left(a_2 m^2(m+1) + \lambda(m^3+3m^2+2m-4)\right)}{m^2\left(a_2 m + \lambda(m+2)\right)^2}\\
&- \frac{2a_1 c_1}{m^4\left(a_2 m + \lambda(m+2)\right)^3}\Big(\left(a_2 m^2(m+1) + \lambda(m^3+3m^2+2m-4)\right)^2 c_1^2\\
&- \left(a_2 m + \lambda(m+2)\right)\left(m^2 c_2\left(a_2 m^2(m+2) + \lambda(m^3+4m^2+4m-8)\right) - 4c_1^2\lambda(m-2)\right)\Big)\\
&- \frac{2a_1 c_3}{\left(a_2 m + \lambda(m+2)\right)} + \frac{1}{12m^9(m+2)}\Big((m+2)\big(3a_3 m^4(m+1)^2(5m+2)\\
&- a_4\lambda(3m^7+18m^6+47m^5+68m^4+96m^3+136m^2+112m+96)\big)c_1^3\\
&- 3m^2(m+2)\left(a_3 m^3(15m^2+26m+8) - a_4\lambda(3m^5+16m^4+36m^3+48m^2+64m+64)\right)c_1 c_2\\
&+ 3m^4\left(3a_3 m^2(5m^2+12m+4) - a_4\lambda(3m^4+18m^3+44m^2+72m+48)\right)c_3\Big)\\
&+ \frac{2a_1}{3m^6\left(a_2 m + \lambda(m+2)\right)^3}\Bigg(-3c_1\left(a_2 m^2(m+1) + \lambda(m^3+3m^2+2m-4)\right)\\
&\times\left(m^2 c_2\left(a_2 m^2(m+2) + \lambda(m^3+4m^2+4m-8)\right) - 4c_1^2(m-2)\lambda\right)\\
&+ \frac{1}{\left(a_2 m + \lambda(m+2)\right)}3c_1\left(a_2 m^2(m+1) + \lambda(m^3+3m^2+2m-4)\right)\\
&\times\Big(\left(a_2 m^2(m+1) + \lambda(m^3+3m^2+2m-4)\right)^2 c_1^2\\
&- \left(a_2 m + \lambda(m+2)\right)\left(c_2 m^2\left(a_2 m^2(m+2) + \lambda(m^3+4m^2+4m-8)\right) - 4c_1^2\lambda(m-2)\right)\Big)\\
&+ \frac{\left(a_2 m + \lambda(m+2)\right)}{(m+2)^2}\Big(4c_1^3\lambda(m+2)^2(m^4+5m^3-4m^2+4m-12) - 12m^2\lambda(m+2)^2(m^2+4m-8)c_1 c_2\\
&+ 3c_3 m^4\left(a_2 m^2(m+2)^2(m+3) + \lambda(m^5+9m^4+30m^3+36m^2-24m-48)\right)\Big)\Bigg)\Bigg]
\end{aligned}$$

with $\lambda = \left(\dfrac{m}{m+2}\right)^m$. In order to get fourth order method, we must have

$$A_1 = A_2 = A_3 = 0$$

solving which, we obtain

$$\begin{aligned}
a_1 &= -\frac{1}{8} m^{m-11}(m+2)^{-4m}\left(a_4 m^m(m+2) - 2m^4(m+2)^m\right)^3,\\
a_2 &= m^{m-5}(m+2)^{-2m}\left(-a_4(m+2)m^m - (m+2)^m m^5\right),\\
a_3 &= \frac{-a_4^2 m^{2m}(m+2)^{2-2m} + a_4 m^{m+4}(m+6)(m+2)^{1-m} - 4(m-2)m^7}{m^5(5m+2)}.
\end{aligned}$$

Also, for these values of the parameters, from (14), the error equation of the method (10) is

$$\begin{aligned}
e_{n+1} = \Big(&\frac{2m^5(m+2)^m\left(3c_3m^5 + c_1(m+2)^2\left(c_1^2(m(m(m+2)+2)-2) - 3c_2m^3\right)\right)}{6m^9(m+2)^{m+2} - 3a_4m^{m+5}(m+2)^3}\\
&- \frac{a_4m^m(m+2)\left(3c_3m^6 + c_1(m+2)^2\left(c_1^2(m(m(m(m+2)+2)-2)+12) - 3c_2m^4\right)\right)}{6m^9(m+2)^{m+2} - 3a_4m^{m+5}(m+2)^3}\Big)e_n^4\\
&+ O(e_n^5)
\end{aligned}$$

and the assertion is proved. □

Method (10) can be modify in such a way that we can also test its convergence for power mean. Precisely, we have

$$\begin{aligned}
y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= x_n - \frac{1}{2}\left[\frac{2a_1 f(x_n)}{a_2 f'(x_n) + f'(y_n)} + \frac{f(x_n)\big(a_3(5m+2)f'(x_n) - (m+2)a_4 f'(y_n)\big)}{4mf'(x_n)^2}\right]\\
&= x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\frac{2a_1}{a_2 + \frac{f'(y_n)}{f'(x_n)}} + \frac{a_3(5m+2) - (m+2)a_4\frac{f'(y_n)}{f'(x_n)}}{4m}\right].
\end{aligned}$$

Now, apply power mean in the second equation, we obtain

$$\begin{aligned}
y_n &= x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\left[\frac{2a_1}{a_2 + \frac{f'(y_n)}{f'(x_n)}} + \frac{a_3(5m+2) - (m+2)a_4\frac{f'(y_n)}{f'(x_n)}}{4m}\right]\\
&= x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{A^p + B^p}{2}\right)^{1/p}, \qquad (15)
\end{aligned}$$

where,

$$\begin{aligned}
A &= \frac{2b_1}{b_2 + \frac{f'(y_n)}{f'(x_n)}},\\
B &= \frac{1}{4m}\left(b_3(5m+2) - (m+2)b_4\frac{f'(y_n)}{f'(x_n)}\right)
\end{aligned}$$

and b_1, b_2 and b_3 are the new parameters. For different values of p in (15), we obtain different methods based on different means. In particular,

For $p = 1$, arithmetic mean.
For $p = -1$, harmonic mean.
For $p = \frac{1}{2}$, sqaure mean root.
For $p \to 0$, geometric mean.

For $p = 1$, it can be proved that method (15) reduces to (10) and hence $b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3$. However, the convergence of the method (15) need to be analyzed for general p.

3 Numerical Examples

In this section, we check the performance of one parameter family of the method (10) with the help of some numerical examples. We denote the method by M_1 for the parameter $a_4 = -1$ and M_2 for the parameter $a_4 = m\left(\dfrac{m}{m+2}\right)^2$. The results obtained are then compared with some existing methods. For this purpose, we take the methods (4), (5) and (6) denoting them by M_3, M_4 and M_5 respectively. Test functions together with their approximate roots x^* upto 16 decimal places and multiplicity m is given in Table 1. Table 2 shows the number of iterations (n) when $|f(x_n)| < 10^{-30}$ and the corresponding absolute value of the function for the methods M_i, $i = 1, 2, 3, 4, 5$. From the table, it is clear that the proposed methods are competitive with other known methods. Here, $a(b)$ represents $a \times 10^{-b}$.

$f(x)$	x^*	m
$f_1(x) = \left(-x^2 + \sin^2(x) + 1\right)^2$	1.4044916482153412	2
$f_2(x) = \left(\exp\left(-x^2\right) + x^2 + x\sin(x) - 2\right)^6$	0.9169529326210010	6
$f_3(x) = (x\exp(x) + \cos(x))^5$	-1.2010606007342120	5
$f_4(x) = (\exp(x) - 2)^2(\cos(x) + 1)$	0.6931471805599453	2
$f_5(x) = (\sqrt{x} + \log(x) - 5)^5$	8.3094326942315718	5

Table 1: Test functions with their approximate roots and multiplicity

$f(x)$	x_0	Methods									
		M_1		M_2		M_3		M_4		M_5	
		n	$f(x_n)$	n	$f(x_n)$	n	$f(x_n)$	n	$f(x_n)$	n	$f(x_n)$
f_1	2.5	3	1.97(31)	3	1.97(31)	4	1.10(31)	3	7.88(31)	3	1.10(31)
	3.5	4	7.88(31)	4	7.88(31)	4	1.10(31)	4	1.10(31)	4	1.10(31)
f_2	1	2	1.28(85)	2	5.59(84)	2	7.67(93)	2	2.41(81)	2	1.13(84)
	1.5	2	3.98(41)	2	3.98(41)	2	1.17(38)	2	5.14(41)	2	4.24(41)
f_3	-2	2	1.03(37)	2	1.03(37)	3	1.86(51)	2	1.02(37)	2	1.03(37)
	1.9	4	4.76(72)	4	4.82(73)	4	3.22(60)	3	1.47(30)	4	1.72(72)
f_4	0	3	7.06(30)	3	3.48(31)		diverge	3	5.58(30)	3	7.85(31)
	1.5	3	0	3	0	3	0	3	1.39(30)	3	0
f_5	1	2	3.06(43)	2	5.24(43)	3	1.77(48)	2	3.83(41)	2	8.03(67)
	10	2	5.65(73)	2	1.81(71)	2	1.09(31)	2	1.09(31)	2	1.34(68)

Table 2: Numerical results obtained from various methods

4 Dynamical analysis

In this section, we study the dynamical behavior of the methods. In particular, we analyze the fixed points, critical points, basins of attraction and stability of the methods presented in this paper. For this, we apply the methods on complex polynomials $p(z)$ with degrees two and three having different multiplicities. It is well known that the fixed points and the critical points of any method play important role in the understanding of the dynamics of the corresponding method. We study the effect of order, degree of the polynomial and multiplicity on the number of extraneous points. Further, stability of the method is shown visually with the help of basins of attraction of the attracting fixed points. In Subsection 4.2, we study and obtain the fixed and critical points of the methods developed in this paper while in 4.3, basins of attraction of the methods are presented. For more details of the complex dynamics of rational functions (or operators), one may refer to [1], [4], [3] and references therein.

4.1 Some basics

Let $\hat{\mathbb{C}} = \mathbb{C} \cup (\infty)$ denote the extended complex plane. Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a function. A point $z_0 \in \hat{\mathbb{C}}$ is called a fixed point of p if $p(z_0) = z_0$. A fixed point z_0 of $p(z)$ is called attracting, super-attracting, repelling or neutral if, respectively, $0 <\mid p'(z_0) \mid< 1$, $p'(z_0) = 0$, $\mid p'(z_0) \mid> 1$, or $\mid p'(z_0) \mid= 1$. It is noted that $z = \infty$ is a super-attracting fixed point for any polynomial with degree $n \geq 2$ [3].

Orbit of a point $z_0 \in \hat{\mathbb{C}}$ of the mapping $p(z)$ is given by

$$orbit(z_0) = \{z_o, p(z_0), p^2(z_0), ...\}$$

The basin of attraction of an attracting (or super-attracting) fixed point z_0 of $p(z)$ is the set of all points whose orbits converge to z_0.

Consider for example the mapping $p(z) = z^2$. It can be checked that 0 and ∞ are super-attracting fixed points while 1 is a repelling fixed point for $p(z)$. Also, the basins of attraction for 0 and ∞ are, respectively $\{z :\mid z \mid< 1\}$and $\{z :\mid z \mid> 1\}$.
Corresponding to the function $p(z)$, define a transform

$$M_p(z) = z - \phi_p(z),$$

where $\phi_p(z)$ is such that $p(z) = 0 \Rightarrow \phi_p(z) = 0$. When $\phi_p(z) = \frac{p(z)}{p'(z)}$, then the corresponding transform $M_p(z)$ is the well known Newton's transform. Clearly, the roots of $p(z) = 0$ are the fixed points of $M_p(z)$. However, there may be fixed points of $M_p(z)$ which need not be the roots of $p(z) = 0$. Such points are called extraneous (or strange) fixed points. Consider, for example, $p(z) = z^2 - 3z + 2$ and

$$M_p(z) = z - \frac{z^2 - 3z + 2}{2z - 3}(z^2 - 3z + 3).$$

Here $z = 1, 2$ are the roots of $p(z) = 0$ and therefore fixed points of $M_p(z)$. The points $z = \frac{3 \pm i\sqrt{3}}{2}$ are not the roots of $p(z) = 0$ but are fixed points of $M_p(z)$ and therefore are the extraneous fixed points of $M_p(z)$.

4.2 Fixed and critical points

Let $p(z)$ be a polynomial having multiple zeros defined on $\hat{\mathbb{C}}$. We define the operator of the method (10) as below:

$$\begin{aligned} y(z) &= z - \frac{2m}{m+2}\frac{p(z)}{p'(z)}, \\ M_i(z) &= z - \frac{1}{2}\left[\frac{2a_1 p(z)}{a_2 p'(z) + p'(y(z)} + \frac{p(z)\left(a_3(5m+2)p'(z) - (m+2)a_4 p'(y(z))\right)}{4mp'(z)^2}\right], \end{aligned} \tag{1}$$

where $i = 1, 2$ and a_1, a_2 and a_3 are as obtained in Theorem (1). We obtain the methods $M_1(z)$ and $M_2(z)$ for the value of the parameter $a_4 = -1$ and $a_4 = m\left(\frac{m}{m+2}\right)^2$ respectively. The fixed points of methods are obtained by $M_i(z) = z$ and critical points are obtained by $M_i'(z) = 0$, $i = 1, 2$. For the second degree and third degree polynomials with different multiplicity the fixed and critical points of the methods are

Polynomial ($p(z)$)	Fixed Points			Critical Points		
	Roots	Number of extraneous points		Roots	Number of free points	
		M_1	M_2		M_1	M_2
$(z^2+1)^2$	i $-i$	8	8	i $-i$	8	8
$(z^2+1)^3$	i $-i$	12	12	i $-i$	14	14
$(z^3-1)^2$	1 $-0.5-0.866025i$ $-0.5+0.866025i$	24	24	1 $-0.5-0.866025i$ $-0.5+0.866025i$	30	30
$(z^3-1)^3$	1 $-0.5-0.866025i$ $-0.5+0.866025i$	36	36	1 $-0.5-0.866025i$ $-0.5+0.866025i$	48	48

Table 3: Fixed and critical points

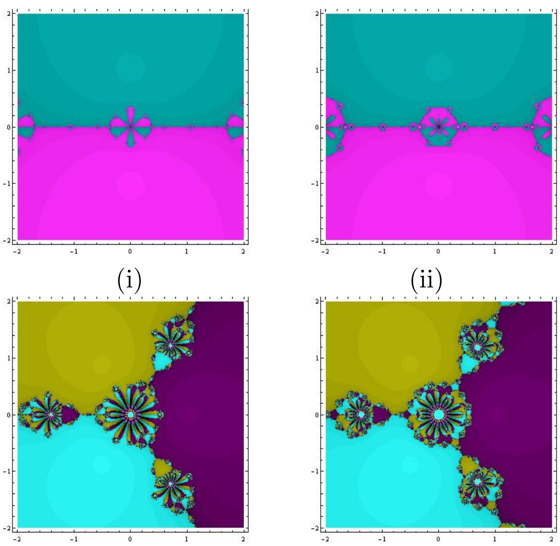
presented in Table 3:

From the table, it is clear that the roots of the polynomials are always the fixed points as well as the critical points. The number of extraneous (or strange) fixed points and number of free critical points varies according to the polynomials used and the multiplicity of any polynomial. However, for any particular polynomial with a given multiplicity, both M_1 and M_2 have same number of fixed as well as critical points. Which shows that the value of the parameter doesn't affect the number of extraneous fixed points and free critical points.

The existence of extraneous fixed points of any operator complicate the root finding procedure. As attractive fixed points, they may trap an iteration sequence, giving erroneous results for a root α of the polynomial $p(z)$. Even as the repulsive or neutral fixed points, however, they may alter the structure of the basin of attraction for the roots [10]. Therefore, large number of extraneous fixed and critical points for the higher degree polynomial make the method less stable.

4.3 Basins of attraction

We describe the dynamical behavior of the methods M_1 and M_2 in terms of their basins of attraction. We do it for second and third degree polynomials with different multiplicities.



(i) (ii)

(iii) (iv)

Figure 1: Basin of attraction of the method $M_1(z)$ for the polynomials: (i) $(z^2+1)^2$, (ii) $(z^2+1)^3$, (iii) $(z^3-1)^2$ and (iv) $(z^3-1)^3$.

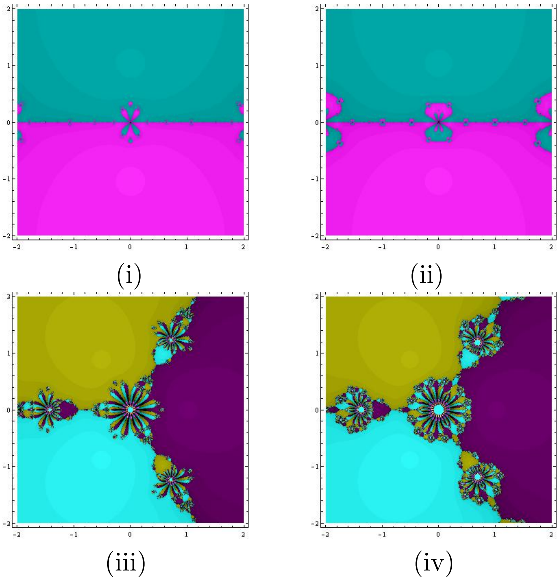


Figure 2: Basin of attraction of the method $M_2(z)$ for the polynomials: (i) $(z^2+1)^2$, (ii) $(z^2+1)^3$, (iii) $(z^3-1)^2$ and (iv) $(z^3-1)^3$.

From Figures 1 and 2, it is clear that basins of attraction of both the methods changes with the change of multiplicities of polynomials. The basins of attraction the methods are more smooth for lower multiplicity $(m = 2)$ as compared to that of higher one $(m = 3)$. It is due to the fact that number of extraneous fixed points and free critical points for the higher multiplicity is more than that of the smaller one.

In our work, we use Mathematica 9.0 for all the numerical calculations as well as to obtain the basins of attraction. We have divided the complex plane into 300×300 initial points in the domain $[-2, 2] \times [-2, 2]$ of the complex plane and use the software of Varona in [9] to determine the basins of attraction of the roots of the polynomials. Light color specifies the region where initial points require less iterations to converge to the particular root. As the color gets darker and darker, it means that the number of iteration increases to converge the root. Black color is used for the initial points which do not converge to any of the root within the maximum limit of 40 iterations.

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