Periodic Components of the Fatou Set of Three Transcendental Entire Functions and Their Compositions

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Abstract: *We prove that there exist three different transcendental entire functions that can have infinite number of domains which lie in the different periodic component of each of these functions and their compositions.*

Keywords: Fatou set, Pre-periodic component, Periodic component, Wandering component, Carleman set
DOI: https://doi.org/10.3126/jnms.v3i1.33002

1 Introduction

We denote the *complex plane* by $\mathbb{C}$, *extended complex plane* by $\mathbb{C}_\infty$ and *set of integers greater than zero* by $\mathbb{N}$. We assume the function $f : \mathbb{C} \to \mathbb{C}$ is *transcendental entire function* unless stated otherwise. For any $n \in \mathbb{N}$, f^n always denotes the nth *iterate* of f. If $f^n(z) = z$ for some smallest $n \in \mathbb{N}$, then we say that z is periodic point of period n. In particular, if $f(z) = z$, then z is a fixed point of f. If $|(f^n)'(z))| < 1$, where $'$ represents complex differentiation of f^n with respect to z, then z is called attracting periodic point of the function f. A family $\mathscr{F} = \{f : f \text{ is meromorphic on some domain } X \text{ of } \mathbb{C}_\infty\}$ forms normal family if every sequence $(f_i)_{i\in\mathbb{N}}$ of functions contains a subsequence which converges uniformly to a finite limit or converges to ∞ on every compact subset D of X.

The *Fatou set* of f denoted by $F(f)$ is the set of points $z \in \mathbb{C}$ such that sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. That is, $z \in F(f)$ if z has a neighborhood U on which the family $\mathscr{F}$ is normal. By definition, Fatou set is open and may or may not be empty. Fatou set is non-empty for every entire function with attracting periodic points. The complement of $F(f)$ denoted by $J(f)$ is called Julia set.

If $U \subset F(f)$ (Fatou component), then $f(U)$ lies in some component V of $F(f)$ and $V - f(U)$ is a set which contains at most one point (see for instance [3]). Let $U \subset F(f)$ (a Fatou component) such that $f^n(U)$ for some $n \in \mathbb{N}$, is contained in some component of $F(f)$, which is usually denoted by U_n. A Fatou component U is called pre-periodic if there exist positive integers n, m such that $U_n = U_m$. In particular, if $U_n = U_0 = U$ (that is, $f^n(U) \subset U$) for some smallest positive integer $n \geq 1$, then U is called *periodic Fatou component* of period n, and $\{U_0, U_1 \ldots, U_{n-1}\}$ is called the *periodic cycle* of U. A component of Fatou set $F(f)$ which is not pre-periodic is called *wandering domain*.

Our particular interest of this paper is that whether there are more than two transcendental entire functions that can have similarity between the dynamics of their compositions and the dynamics of each of these functions. Dynamics of two transcendental entire functions and their compositions were studied by Singh [6]. He constructed several examples of transcendental entire functions where the dynamics of individual functions is similar to the dynamics of their compositions. In the same paper, he also constructed several examples where the dynamics of individual functions vary largly from the dynamics of their compositions. In particular, Singh proved that there exists a domain which lies in the periodic component of individual functions and also lies in the periodic component of the one of the composition but lies in the wandering component of the other compositions (Theorem 4). Later, Kumar et al. [4] extended this result to the possibility of having infinitely many domains satisfying the condition of Singh's result. In [7, Theorem 1.1] and [8, Theorem 1.1] we investigated three different transcendental entire functions such that wandering and pre-periodic components of the Fatou set of each of these functions and their compositions contain infinitely many domains. In this paper, we investigate three different transcendental entire functions such that each of these individual functions as well as their every possible composition with each other, consists of infinite number of domains which lie in the periodic component of each of functions and also in their compositions. In particular, we prove the following assertion.

Theorem 1.1. *There exist three different transcendental entire functions f, g and h and infinite number of domains which lie in the different periodic component of $F(f), F(g), F(h)$, $F(f\circ g), F(g\circ f), F(f\circ h), F(g\circ h), F(h\circ f), F(h\circ g), F(f\circ g\circ h), F(f\circ h\circ g), F(g\circ f\circ h), F(g\circ h\circ f), F(h\circ f\circ g)$ and $F(h\circ g\circ f)$.*

2 Approximation of Holomorphic Functions

We prove Theorem 1.1 by using the notion of Carleman set from which we obtain approximation of any holomorphic function by entire functions

Definition 2.1 (**Carleman Set**). *Let F be a closed subset of $\mathbb{C}$ and*

$$C(F) = \{f : F \to \mathbb{C} : f \text{ is continuous on } F \text{ and analytic in the interior of } F^\circ \text{ of } F\}.$$

Then F is called a Carleman set for $\mathbb{C}$ if for any $g \in C(F)$ and any positive continuous function ϵ on F, there exists entire function h such that $|g(z) - h(z)| < \epsilon$ for all $z \in F$.

The following important characterization of Carleman set proved by Nersesjan in 1971 but we have cited this from [2, Theorem 4, page 157]]

Theorem 2.1. *Let F be a proper subset of $\mathbb{C}$. Then F is a Carleman set for $\mathbb{C}$ if and only if F satisfies the following conditions:*

1. *$\mathbb{C}_\infty - F$ is connected;*
2. *$\mathbb{C}_\infty - F$ is locally connected at ∞;*
3. *for every compact subset K of $\mathbb{C}$, there is a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of F° intersects both K and V.*

It is well known in classical complex analysis that the space $\mathbb{C}_\infty - F$ is connected if and only if each component Z of open set $\mathbb{C} - F$ is unbounded. This fact together with Theorem 2.1 can be a nice tool for checking whether a set is a Carleman set for $\mathbb{C}$. The sets given in the following examples are Carlemen sets for $\mathbb{C}$.

Example 2.1 ([2, Example page 133]). *The set $E = \{z \in \mathbb{C} : |z| = 1, Rez > 0\} \cup \{z = x : x > 1\} \cup \left(\bigcup_{n=3}^{\infty}\{z = re^{i\theta} : r > 1, \theta = \pi/n\}\right)$ is a Carleman set for $\mathbb{C}$.*

Example 2.2 ([6, Set S, page 131]). *The set $E = G_0 \cup \left(\bigcup_{k=1}^{\infty}(G_k \cup B_K \cup L_k \cup M_k)\right)$, where*

$$G_0 = \{z \in \mathbb{C} : |z - 2| \le 1\};$$

$$\begin{aligned} G_k &= \{z \in \mathbb{C} : |z - (4k+2)| \le 1\} \cup \{z \in \mathbb{C} : Rez = 4k+2,\ Imz \ge 1\} \\ &\quad \cup \{z \in \mathbb{C} : Rez = 4k+2,\ Imz \le -1\}, \quad (k = 1, 2, 3, \ldots); \end{aligned}$$

$$\begin{aligned} B_k &= \{z \in \mathbb{C} : |z + (4k+2)| \le 1\} \cup \{z \in \mathbb{C} : Rez = -(4k+2),\ Imz \ge 1\} \cup \\ &\quad \{z \in \mathbb{C} : Rez = -(4k+2),\ Imz \le -1\}, \quad (k = 1, 2, 3, \ldots); \end{aligned}$$

$$L_k = \{z \in \mathbb{C} : Rez = 4k\}, \quad (k = 1, 2, 3, \ldots);$$

and

$$M_k = \{z \in \mathbb{C} : Rez = -4k\}, \quad (k = 1, 2, 3, \ldots)$$

is a Carleman set for $\mathbb{C}$ by Theorem 2.1.

3 Proof of Theorem 1.1

If f and g are two transcendental entire functions, so are their compositions $f \circ g$ and $g \circ f$, and the dynamics of one composition may help in the study of the dynamics of the other composition as shown in the following result Bergweiler and Wang [1].

Proposition 3.1. *Let f and g be two transcendental entire functions. Then $f \circ g$ has wandering domains if and only if $g \circ f$ has wandering domains.*

It should also be noted that certain classes of entire functions do not have wandering domains (see for instance [1, Theorem 3]). Singh [6] interested to know whether there is similarity between dynamics of individual entire functions and their compositions. However, in reality, it does not hold in general. From the help of the Carleman set of Example 2.2, Singh [6, Theorem 4] proved the following result.

Proposition 3.2. *There exist transcendental entire functions f and g and a domain which lies in the periodic component of $F(f)$ and periodic component of $F(g)$ and also in the periodic component of $F(g \circ f)$ but lies in the wandering component of $F(f \circ g)$.*

In fact, Singh [6] also proved other results regarding the dynamics of two individual functions and their compositions (see for instance [6, Theorems 1, 2 and 3]) which are stricly based on the Carleman set of Example 2.2. Kumar et al. [4, Theorem 2.3] extended this result to the following assertion.

Proposition 3.3. *There exists transcendental entire functions f and g having infinite number of domains which lie in periodic component of $F(f)$ and periodic component of $F(g)$ and also lie in the periodic component of $F(g \circ f)$ but lies in the wandering component of $F(f \circ g)$.*

Tomar [9] extended these results within an angular region to the following results.

Proposition 3.4. *There exist two transcendental entire functions f and g and infinitely many domains in the angular region which lies in the periodic component of $F(f)$ and periodic component of $F(g)$ but in the wandering component of $F(f \circ g)$ and wandering component of $F(g \circ f)$.*

Proposition 3.5. *There exist two transcendental entire functions f and g and infinitely many domains in the angular region which lies in the wandering component of $F(f)$ and wandering component of $F(g)$ and also in the wandering component of $F(f \circ g)$ and wandering component of $F(g \circ f)$. Also, simultaneously for the same functions f and g, there exists infinite number of domains which lies in the periodic component of $F(f)$, periodic component of $F(g)$ and also in the periodic component of $F(f \circ g)$ and periodic component of $F(g \circ f)$.*

Note that Singh [5, Theorems 3.2.1 - 3.2.6] studied different components of the Fatou set of a transcendental entire function in an angular region by using approximation theory of entire functions, in particular, by the help of Carleman set.

We extended similar type of results (in the case of wandering and pre-periodic components) of Proposition 3.5 in [7, 8, Theorem 1.1]. Theorem 1.1 is an extension of Proposition 3.5 in the case of periodic components. In particular, Proposition 3.5 can be extended to the existence of more than two different transcendental entire functions such that each individual functions and their compositions may have infinitely many domains which lie in different periodic component of each of the functions and their compositions. We proceed for the following long proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$E = G_0 \cup \Big(\bigcup_{k=1}^{\infty} (G_k \cup B_K \cup L_k \cup M_k) \Big)$$

where G_0, G_k, B_k, L_k and M_k are sets as defined in Example 2.2. Then E is a Carleman set for $\mathbb{C}$. By the continuity of exponential map, for a given $\epsilon > 0$, there exists $\delta > 0$ may, depend on a given point w_0, such that

$$|w - w_0| < \delta \Longrightarrow |e^w - e^{w_0}| < \epsilon.$$

If $w_0 = \log t$, then $e^{w_0} = e^{\log t} = t$. Let us choose $\epsilon = 1/2$, there exist sufficiently small $\delta_{k_1} > 0,\ \delta_{k_2} > 0, \delta^{'}_{k_1} > 0,\ \delta^{'}_{k_2} > 0, \delta_{k_3} > 0$ and $\delta^{'}_{k_3} > 0$ such that

$$|w - \log(4k+6)| < \delta_{k_1} \Longrightarrow |e^w - (4k+6)| < 1/2, \ \ (k = 1, 3, 5, \ldots)$$

$$|w - \log(4k-2)| < \delta_{k_2} \Longrightarrow |e^w - (4k-2)| < 1/2, \ \ (k = 2, 4, 6, \ldots)$$

$$|w - (\pi i + \log(4k+6))| < \delta^{'}_{k_1} \Longrightarrow |e^w + (4k+6)| < 1/2, \ \ (k = 1, 3, 5, \ldots)$$

$$|w - (\pi i + \log(4k-2))| < \delta^{'}_{k_2} \Longrightarrow |e^w + (4k-2)| < 1/2, \ \ (k = 2, 4, 6 \ldots)$$

$$|w - \log(4k+2)| < \delta_{k_3} \Longrightarrow |e^w - (4k+2)| < 1/2, \ \ (k = 1, 2, 3, \ldots)$$

and

$$|w - (\pi i + \log(4k+2))| < \delta^{'}_{k_3} \Longrightarrow |e^w + (4k+2)| < 1/2, \ \ (k = 1, 2, 3, \ldots)$$

In particular, let us choose sufficiently small $\delta_0 > 0,\ \delta_1 > 0$ and $\delta^{'}_1 > 0$ such that

$$|w - \log 2| < \delta_0 \Longrightarrow |e^w - 2| < 1/2.$$

$$|w - \log 6| < \delta_1 \Longrightarrow |e^w - 6| < 1/2.$$

and

$$|w - (\pi i + \log 6)| < \delta^{'}_1 \Longrightarrow |e^w + 6| < 1/2.$$

Next, let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right) \\ \log(4k+2), & \forall z \in B_k,\ k = 1, 2, 3, \ldots \\ \pi i + \log(4k+2), & \forall z \in G_k,\ k = 1, 2, 3, \ldots \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right) \\ \log(4k+6), & \forall z \in B_k,\ k = 1, 3, 5, \ldots \\ \log(4k-2), & \forall z \in B_k,\ k = 2, 4, 6, \ldots \\ \pi i + \log(4k+6), & \forall z \in G_k,\ k = 1, 3, 5, \ldots \\ \pi i + \log(4k-2), & \forall z \in G_k,\ k = 2, 4, 6, \ldots \end{cases}$$

$$\gamma(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right) \\ \log(4k+6), & \forall z \in G_k,\ k = 1, 2, 3, \ldots, n-1; \\ \log 6, & \forall z \in G_n, \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1, 2, 3, \ldots, n-1; \\ \pi i + \log 6, & \forall z \in B_n, \end{cases}$$

$$\epsilon_1(z) = \begin{cases} \delta, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right) \\ \delta_{k_3}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots \\ \delta^{'}_{k_3}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots \end{cases}$$

$$\epsilon_2(z) = \begin{cases} \delta, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right) \\ \delta_{k_1}, & \forall z \in B_k,\ k = 1, 3, 5 \ldots \\ \delta_{k_2}, & \forall z \in B_k,\ k = 2, 4, 6 \ldots \\ \delta^{'}_{k_1}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots \\ \delta^{'}_{k_2}, & \forall z \in G_k,\ k = 2, 4, 6 \ldots \end{cases}$$

and

$$\epsilon_3(z) = \begin{cases} \delta, & \forall z \in G_0 \cup \bigcup_{k=1}^{\infty}(B_K \cup L_k \cup M_k) \\ \delta_{k_1}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots, n-1; \\ \delta_1, & \forall z \in G_n, \\ \delta^{'}_{k_1}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots, n-1; \\ \delta^{'}_1, & \forall z \in B_n, \end{cases}$$

Clearly, the functions $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are piece wise constant functions, so they are continuous on the set E and holomorphic in E°. Also, since E is a Carleman set, so there exist an entire functions $f_1(z)$, $g_1(z)$ and $h_1(z)$ (say) such that

$$\forall z \in E,\ \ |f_1(z) - \alpha(z)| \leq \epsilon_1(z),\ |g_1(z) - \beta(z)| \leq \epsilon_2(z) \text{ and } |h_1(z) - \gamma(z)| \leq \epsilon_3(z).$$

Consequently, we get transcendental entire functions $f(z) = e^{f_1(z)}$, $g(z) = e^{g_1(z)}$ and $h(z) = e^{h_1(z)}$ which respectively satisfy the following:

$$\begin{array}{ll} |f(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |f(z) - (4k+2)| < 1/2, & \forall z \in B_k,\ k = 1, 2, 3 \ldots; \\ |f(z) + (4k+2)| < 1/2, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.1}$$

$$\begin{array}{ll} |g(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |g(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k = 1, 3, 5, \ldots; \\ |g(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k = 2, 4, 6, \ldots; \\ |g(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k = 1, 3, 5, \ldots; \\ |g(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k = 2, 4, 6, \ldots; \end{array} \tag{3.2}$$

and

$$\begin{array}{ll} |h(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |h(z) - (4k+6)| < 1/2, & \forall z \in G_k, k = 1, 2, 3, \ldots, n-1; \\ |h(z) - 6| < 1/2, & \forall z \in G_n; \\ |h(z) + (4k+6)| < 1/2, & \forall z \in B_k, k = 1, 2, 3, \ldots, n-1; \\ |h(z) + 6| < 1/2, & \forall z \in B_n; \\ |h(z) - (4k+2)| < 1/2, & \forall z \in B_k,\ k > n; \\ |h(z) + (4k+2)| < 1/2, & \forall z \in G_k,\ k > n; \end{array} \tag{3.3}$$

From (3.1), (3.2), and (3.3), we can conclude that each of the functions f, g and h maps the domain $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ into smaller disk $|z - 2| < 1/2$ contained in G_0 and each of these function is a contracting mapping. So, $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ contain fixed points z_1, z_2 and z_3 (say) such that

$$f^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_1 \text{ as } n \longrightarrow \infty.$$

$$g^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_2 \text{ as } n \longrightarrow \infty.$$

$$h^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_3 \text{ as } n \longrightarrow \infty.$$

The fixed points z_1, z_2 and z_3 are respectively the attracting fixed points for the functions f, g and h, so $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ lies in attracting cycle and hence $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ is a subset of each of the Fatou set $F(f), F(g)$ and $F(h)$. $J(f) \neq \mathbb{C}$, $J(g) \neq \mathbb{C}$ and $J(h) \neq \mathbb{C}$ and so Julia set of each of the function f, g and h does not contain interior point and hence Fatou set of each of these function contains all interior points. Fatou set of each of the function f, g and h contains Carleman set E.

Again, from (3.1), we can say that function f maps each G_k into smaller disk contained in B_k and B_k into smaller disk contained in G_k for each $k \in \mathbb{N}$. In fact, G_k and B_k are periodic components of period 2 of the function f. Since f is a contraction mapping, so each domain G_k and B_k, $(k = 1, 2, 3, \ldots)$ are periodic component of period 2 that lies in periodic components of $F(f)$. And that of from (3.2), we can say that function g maps each of the domains G_k and B_k $(k = 1, 2, 3, \ldots)$ respectively into the smaller disks of B_k and G_k and each of these domain are also periodic component of period 2 and so they lie in the periodic component of the Fatou set $F(g)$. Likewise, from (3.3), we can say that domains G_k and $B_k, (k \leq n)$ are periodic components of period n under the function h and G_k and B_k are periodic component (of period 2 for $k > n$ and so all these domains lie in periodic component of Fatou set $F(h)$.

Next, we examine the dynamical behavior of compositions of the functions f, g and h. The composition of any two and all three of these functions satisfy the following:

Dynamical behavior of $f \circ g$:

$$\begin{array}{ll} |(f\circ g)(z)-2|<1/2, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right); \\ |(f\circ g)(z)+(4k+6)|<1/2, & \forall z\in B_k,\ k=1,3,5,\ldots; \\ |(f\circ g)(z)+(4k-2)|<1/2, & \forall z\in B_k,\ k=2,4,6,\ldots; \\ |(f\circ g)(z)-(4k+6)|<1/2, & \forall z\in G_k,\ k=1,3,5,\ldots; \\ |(f\circ g)(z)-(4k-2)|<1/2, & \forall z\in G_k,\ k=2,4,6,\ldots; \end{array} \tag{3.4}$$

This composition rule (3.4) shows that the domains $G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right)$, G_k and $B_k,\ (k=1,2,3,\ldots)$ belong to $F(f\circ g)$ and in fact, each G_k and B_k for each $k\in\mathbb{N}$ is a periodic domain of period 2 which belongs to the periodic components of $F(f\circ g)$.

Dynamical behavior of $g \circ f$:

$$\begin{array}{ll} |(g\circ f)(z)-2|<1/2, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right); \\ |(g\circ f)(z)+(4k+6)|<1/2, & \forall z\in B_k,\ k=1,3,5,\ldots; \\ |(g\circ f)(z)+(4k-2)|<1/2, & \forall z\in B_k,\ k=2,4,6,\ldots; \\ |(g\circ f)(z)-(4k+6)|<1/2, & \forall z\in G_k,\ k=1,3,5,\ldots; \\ |(g\circ f)(z)-(4k-2)|<1/2, & \forall z\in G_k,\ k=2,4,6,\ldots; \end{array} \tag{3.5}$$

From this composition rule (3.5), we can say that the domains $G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right)$, G_k and $B_k,\ (k=1,2,3,\ldots)$ belong to $F(g\circ f)$ and in fact, each G_k and B_k for each $k\in\mathbb{N}$ is a periodic domain of period 2 which belongs to the periodic components of $F(g\circ f)$.

Dynamical behavior of $f \circ h$:

$$\begin{array}{ll} |(f\circ h)(z)-2|<1/2, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right); \\ |(f\circ h)(z)-(4k+6)|<1/2, & \forall z\in B_k,\ k=1,2,3,\ldots,n-1; \\ |(f\circ h)(z)-6|<1/2, & \forall z\in B_n; \\ |(f\circ h)(z)+(4k+6)|<1/2, & \forall z\in G_k,\ k=1,2,3,\ldots,n-1; \\ |(f\circ h)(z)+6|<1/2, & \forall z\in G_n; \\ |(f\circ h)(z)+(4k+2)|<1/2, & \forall z\in B_k,\ k>n; \\ |(f\circ h)(z)-(4k+2)|<1/2, & \forall z\in G_k, k>n; \end{array} \tag{3.6}$$

The composition rule (3.6) shows the domains $G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right)$, G_k and $B_k,\ (k=1,2,3,\ldots)$ belong to $F(f\circ h)$. In fact, each G_k and $B_k,\ (k>n)$ is a periodic domain of period 1 and each G_k and B_k for $k=1,2,3,\ldots,n$ is a periodic component of period n for even n.

Dynamical behavior of $h \circ f$:

$$\begin{array}{ll} |(h\circ f)(z)-2|<1/2, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right); \\ |(h\circ f)(z)+(4k+6)|<1/2, & \forall z\in B_k,\ k=1,2,3,\ldots,n-1; \\ |(h\circ f)(z)+6|<1/2, & \forall z\in B_n; \\ |(h\circ f)(z)-(4k+6)|<1/2, & \forall z\in G_k,\ k=1,2,3\ldots,n-1; \\ |(h\circ f)(z)-6|<1/2, & \forall z\in G_n; \\ |(h\circ f)(z)+(4k+2)|<1/2, & \forall z\in B_k,\ k>n; \\ |(h\circ f)(z)-(4k+2)|<1/2, & \forall z\in G_k, k>n; \end{array} \tag{3.7}$$

From this composition rule (3.7), we can say that the domains $G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right)$, G_k and $B_k,\ (k=1,2,3,\ldots)$ belong to $F(h\circ f)$. In fact, each G_k and $B_k,\ (k>n)$ is a periodic component of period 1 and each G_k and B_k for $k=1,2,3,\ldots,n$ is a periodic component of period n for odd n.

Dynamical behavior of $g \circ h$:

$$
\begin{array}{ll}
|(g\circ h)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
|(g\circ h)(z) - (4k+2)| < 1/2, & \forall z \in B_k,\ k = 1,3,5\ldots,n-1;\\
|(g\circ h)(z) - (4k+10)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1;\\
|(g\circ h)(z) - 10| < 1/2, & \forall z \in B_n;\\
|(g\circ h)(z) + (4k+2)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1;\\
|(g\circ h)(z) + (4k+10)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1;\\
|(g\circ h)(z) + 10| < 1/2, & \forall z \in G_n;\\
|(g\circ h)(z) + (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even};\\
|(g\circ h)(z) + (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd};\\
|(g\circ h)(z) - (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even};\\
|(g\circ h)(z) - (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd};
\end{array}
\tag{3.8}
$$

This composition rule (3.8) shows that the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1,2,3,\ldots)$ belong to $F(g\circ h)$. In fact, each G_k and $B_k,\ (k > n)$ is a periodic components of of period 1 and each G_k and B_k for odd $k \leq n$ are periodic components of period 2.

Dynamical behavior of $h \circ g$:

$$
\begin{array}{ll}
|(h\circ g)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
|(h\circ g)(z) - (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1,3,5\ldots,n-1;\\
|(h\circ g)(z) - (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1;\\
|(h\circ g)(z) + (4n+6)| < 1/2, & \forall z \in B_n,\ \text{for odd } n;\\
|(h\circ g)(z) - (4n+2)| < 1/2, & \forall z \in B_n,\ \text{for even } n;\\
|(h\circ g)(z) + (4k+10)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1;\\
|(h\circ g)(z) + (4k+2)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1;\\
|(h\circ g)(z) - (4n+6)| < 1/2, & \forall z \in G_n,\ \text{for odd } n;\\
|(h\circ g)(z) + (4n+2)| < 1/2, & \forall z \in G_n,\ \text{for even } n;\\
|(g\circ h)(z) + (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even};\\
|(h\circ g)(z) - 6| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k-1 = n;\\
|(h\circ g)(z) + (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd};\\
|(h\circ g)(z) - (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even};\\
|(h\circ g)(z) + 6| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k-1 = n;\\
|(h\circ g)(z) - (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd};
\end{array}
\tag{3.9}
$$

From this composition rule (3.9), we can say that the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1,2,3,\ldots)$ belong to $F(h\circ g)$. In fact, each G_k and $B_k,\ (k > n)$ is a periodic components of period 1 and each G_k and B_k for even $k \leq n$ is a periodic component of period 2.

Dynamical behavior of $f \circ g \circ h$:

$$
\begin{array}{ll}
|(f\circ g\circ h)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
|(f\circ g\circ h)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\ldots,n-1;\\
|(f\circ g\circ h)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1;\\
|(f\circ g\circ h)(z) + 10| < 1/2, & \forall z \in B_n;\\
|(f\circ g\circ h)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1;\\
|(f\circ g\circ g)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1;\\
|(f\circ g\circ g)(z) - 10| < 1/2, & \forall z \in G_n;\\
|(f\circ g\circ h)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd};\\
|(f\circ g\circ h)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even};\\
|(f\circ g\circ h)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd};\\
|(f\circ g\circ h)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even};
\end{array}
\tag{3.10}
$$

The composition rule (3.10) assigned above tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1,2,3,\ldots)$ lie in $F(f\circ g\circ h)$. In fact, each G_k and $B_k,\ (k > n)$ is a periodic components of period 1 and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $f \circ h \circ g$:

$$\begin{array}{ll}
|(f \circ h \circ g)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(f \circ h \circ g)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\ldots,n-1; \\
|(f \circ h \circ g)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1; \\
|(f \circ h \circ g)(z) + (4n+6)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is odd}; \\
|(f \circ h \circ g)(z) + (4n+2)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is even}; \\
|(f \circ h \circ g)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1; \\
|(f \circ h \circ g)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1; \\
|(f \circ h \circ g)(z) + (4n+6)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is odd}; \\
|(f \circ h \circ g)(z) - (4n+2)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is even}; \\
|(f \circ h \circ g)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd}; \\
|(f \circ h \circ g)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even}; \\
|(f \circ h \circ g)(z) + 6| < 1/2, & \forall z \in B_k,\ k > n\ k \text{ is even and } k-1 = n; \\
|(f \circ h \circ g)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd}; \\
|(f \circ h \circ g)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even}; \\
|(f \circ h \circ g)(z) - 6| < 1/2, & \forall z \in G_k,\ k > n\ k \text{ is even and } k-1 = n;
\end{array} \tag{3.11}$$

The composition rule (3.11) assigned above tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1,2,3,\ldots)$ lie in $F(f \circ h \circ g)$ and in fact, each G_k and B_k, $(k > n)$ belong to the periodic components of $F(f \circ h \circ g)$ of period 1. Each G_k and B_k for even k are periodic components of period 1.

Dynamical behavior of $g \circ f \circ h$:

$$\begin{array}{ll}
|(g \circ f \circ h)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(g \circ f \circ h)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\ldots,n-1; \\
|(g \circ f \circ h)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1; \\
|(g \circ f \circ h)(z) + 10| < 1/2, & \forall z \in B_n; \\
|(g \circ f \circ h)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1; \\
|(g \circ f \circ h)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1; \\
|(g \circ f \circ h)(z) - 10| < 1/2, & \forall z \in G_n; \\
|(g \circ f \circ h)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd}; \\
|(g \circ f \circ h)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even}; \\
|(g \circ f \circ h)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd}; \\
|(g \circ f \circ h)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even};
\end{array} \tag{3.12}$$

The composition rule (3.12) tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1,2,3,\ldots)$ lie in $F(g \circ f \circ h)$. In fact, each G_k and B_k, $(k > n)$ is a periodic components of period 1 and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $g \circ h \circ f$:

$$\begin{array}{ll}
|(g \circ h \circ f)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(g \circ h \circ f)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\ldots,n-1; \\
|(g \circ h \circ f)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\ldots,n-1; \\
|(g \circ f \circ h)(z) + 10| < 1/2, & \forall z \in B_n; \\
|(g \circ h \circ f)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\ldots,n-1; \\
|(g \circ g \circ f)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\ldots,n-1; \\
|(g \circ f \circ h)(z) - 10| < 1/2, & \forall z \in G_n; \\
|(g \circ h \circ f)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd}; \\
|(g \circ h \circ f)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even}; \\
|(g \circ h \circ f)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd}; \\
|(g \circ h \circ f)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even};
\end{array} \tag{3.13}$$

The composition rule (3.13) tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1,2,3,\ldots)$ lie in $F(g \circ h \circ f)$. In fact, each G_k and B_k, $(k > n)$ is a periodic components of period 1 and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1. Note that composition rules 3.12 and 3.13 show that dynamics of $g \circ f \circ h$ and $g \circ h \circ f$ coincide.

Dynamical behavior of $h \circ f \circ g$:

$$\begin{array}{ll}
|(h \circ f \circ g)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(h \circ f \circ g)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1, 3, 5, \ldots, n-1; \\
|(h \circ f \circ g)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2, 4, 6, \ldots, n-1; \\
|(h \circ f \circ g)(z) - (4n+6)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is odd}; \\
|(h \circ f \circ g)(z) + (4n+2)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is even}; \\
|(h \circ f \circ g)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 1, 3, 5 \ldots, n-1; \\
|(h \circ g \circ g)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 2, 4, 6, \ldots, n-1; \\
|(h \circ f \circ g)(z) + (4n+6)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is odd}; \\
|(h \circ f \circ g)(z) - (4n+2)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is even}; \\
|(h \circ f \circ g)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd}; \\
|(h \circ f \circ g)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even}; \\
|(h \circ f \circ g)(z) + 6| < 1/2, & \forall z \in B_k,\ k > n\ k \text{ is even and } k-1 = n; \\
|(h \circ f \circ g)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd}; \\
|(h \circ f \circ g)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even}; \\
|(h \circ f \circ g)(z) - 6| < 1/2, & \forall z \in G_k,\ k > n\ k \text{ is even and } k-1 = n;
\end{array} \tag{3.14}$$

The composition rule (3.14) tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(h \circ f \circ g)$. In fact, each G_k and $B_k,\ (k > n)$ is a periodic component of period 1 and each G_k and B_k for even $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $h \circ g \circ f$:

$$\begin{array}{ll}
|(h \circ g \circ f)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(h \circ g \circ f)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1, 3, 5, \ldots, n-1; \\
|(h \circ g \circ f)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2, 4, 6, \ldots, n-1; \\
|(h \circ g \circ f)(z) - (4n+6)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is odd}; \\
|(h \circ g \circ f)(z) + (4n+2)| < 1/2, & \forall z \in B_n \text{ and } n \text{ is even}; \\
|(h \circ g \circ f)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 1, 3, 5 \ldots, n-1; \\
|(h \circ g \circ f)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 2, 4, 6, \ldots, n-1; \\
|(h \circ g \circ f)(z) + (4n+6)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is odd}; \\
|(h \circ g \circ f)(z) - (4n+2)| < 1/2, & \forall z \in G_n \text{ and } n \text{ is even}; \\
|(h \circ g \circ f)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is odd}; \\
|(h \circ g \circ f)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and } k \text{ is even}; \\
|(h \circ g \circ f)(z) + 6| < 1/2, & \forall z \in B_k,\ k > n\ k \text{ is even and } k-1 = n; \\
|(h \circ g \circ f)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is odd}; \\
|(h \circ g \circ f)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and } k \text{ is even}; \\
|(h \circ g \circ f)(z) - 6| < 1/2, & \forall z \in G_k,\ k > n\ k \text{ is even and } k-1 = n;
\end{array} \tag{3.15}$$

The composition rule (3.15) tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(h \circ g \circ f)$. In fact, each G_k and $B_k,\ (k > n)$ is a periodic components of period 1 and each G_k and B_k for even $k \leq n$ is a periodic component of period 1.

From all of the above discussion, we found that the domain $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ and all domains G_k and $B_k,\ (k > n)$ are periodic components period 2 for the composites of any two functions and period 1 for the composites of three functions. So all, these domains lie in the Fatou sets of the functions f, g and h and their compositions. Also, there are other periodic domains G_k and B_k for $k \leq n$ of different periods of the composites that lie in the periodic components of the Fatou sets the functions f, g and h and their compositions. □

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