A New Postulate and Some Conjectures Concerning Pair Primes in the Interval $[n!, (n+k)!]$

Abiodun E. Adeyemi

Department of Mathematics, University of Ibadan Ibadan, Oyo state, Nigeria

Correspondence to: Abiodun E. Adeyemi, Email: elijahjje@yahoo.com

Abstract: *This paper rather studies the behaviour of prime numbers bounded below and above by positive integers* $n!$ *and* $(n+k)!$*, and then after some numerical evidence, postulates that there is at least one pair primes of gap* $k \in 2\mathbb{Z}^+$ *in between* $n!$ *and* $(n+k)!$ *for every integer* $n \geq 2$ *and every even integer* $k > 0$*. This assertion would eventually provide another structural form for Euclid theorem of infinitude of primes, a kind of projection of the form in the original Bertrand postulate (now Chebychev's theorem). The truthfulness of the conjecture that emanated from this postulate implies the Polignac's conjecture which aptly generalizes the twin prime conjecture. We thus present the new postulate and the conjectures for future research.*

Keywords: Primes, Pair primes, Bertrand postulate
DOI: https://doi.org/10.3126/jnms.v3i1.32996

1 Introduction

To start with, the study of prime numbers is quite interesting, deeply challenging, and really applicable, as could be inferred from the following quotes:

> "Some numbers, even large ones, have no factors - except themselves, of course, and 1. These are called prime numbers, because every thing they are starts with themselves. They are original, gnarled, unpredictable ..." Friedberg in his Adventurer's guide to number theory.

> "To some extent, the beauty of number theory seems to be related to the contradiction between the simplicity of the integers and the complicated structure of the primes, their building blocks..." Knauf in his 1998 Lecture notes on Number Theory, Dynamical systems and statistical mechanics.

> "Prime numbers are the most basic objects in mathematics. They are also among the most mysterious, for after centuries of study, the structure of the set of prime numbers is still not well understood. Describing the distribution of primes is at the heart of much mathematics and has many important applications to such areas as cryptography..." Granville in December 5, 1997 AMS News.

In a nutshell, according to Crandal and Pomerance [2],

> "Prime numbers belong to an exclusive world of intellectual conceptions. We speak of those marvellous notions that enjoy simple, elegant description, yet lead to extreme - one might say unthinkable - complexity in the details".

In concrete mathematical terms, as at the present moment, concerning primes $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ p_6 = 13,\ p_7 = 17,\ p_8 = 19,\ p_9 = 23,\ p_{10} = 29,\ ..., p,\ p_n \in \mathbb{P}$ the known results are the consequences and improvement of the following assertions:

$$p \mid \prod_{i=1}^{n} p_i + 1 \Rightarrow p \notin \{p_1, p_2, ...p_n \ \forall\ n\} \qquad (\textit{Euclid's theorem})[3],$$

meaning that any prime number p that divides the product of the preceding primes plus 1 is always a new prime number, since every integer must have a prime factor, a consequence of the fundamental theorem of arithmetic; thus, prime numbers keep coming i.e infinitely many are prime numbers.

$$n = \prod_{i=1}^{\omega(n)} p_i^{n_i} \qquad \textit{(Unique prime factorization theorem)}[7],$$

which states that every positive integer n can be uniquely expressed as a product of distinct prime numbers.

$$\pi(x) \approx \frac{x}{\ln x} \qquad \textit{(Prime number theorem)}[3],$$

with the interpretation that the number of prime numbers that do not exceed real x (denoted by $\pi(x)$) can be obtained by computing $x/\ln x$ (another function) though with a little error which diminishes as the bound x approaches infinity. The beauty of this formular is encapsulated in the following letter of Gauss to Encke in 1849,

> *"As a boy I considered the problem of how many primes there are up to a given point. From my computations, I determined that the density of primes around x, is about $1/\log x$."*[3]

$$\pi(x) \approx Li(x) = \int_2^x \frac{1}{\ln t} dt \qquad \textit{(Prime number theorem by Peter Gustav[3])},$$

this reveals that the number of prime numbers that do not exceed x can better be gotten by computing $\int_2^x \frac{1}{\ln t} dt$, although the result is not without a little error which diminishes as the bound x approaches infinity.

$$\zeta(s) = \sum \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1-p^{-s}} \ (\Re(s) > 1) \qquad \textit{(Euler product form for Dirichilet's series)}[3,\ 7],$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \in \mathbb{P}, m \geq 1} \frac{\ln p}{p^{ms}} \ (\Re(s) > 1) \qquad \textit{(Euler product form for Dirichilet's series)}[3,\ 7],$$

signifying that prime numbers are connected to the Riemann zeta function ($\zeta(s)$) which is the sum of reciprocal of every natural number n of exponent s, a generalization of Drichilet's series to complex numbers s.

$$\sum_{p \in \mathbb{P}} \frac{1}{p} \ \textit{diverges} \qquad \textit{(Euler theorem)}, [2],$$

$$\sum_{p, p+2 \in \mathbb{P}} \frac{1}{p} \ \textit{converges} \qquad \textit{(Brun's theorem)}[2],$$

revealing that the appearance of twin primes (pair primes with gap of 2) are not as prevalent as infinite primes that appear in the set of natural numbers.

$$p_{n+1} \leq 2p_n \qquad \textit{(Bertrand postulate)}[2],$$

an assertion that the next prime is not as much as twice the preceding one.

$$\lim_{n\to\infty} \frac{\pi(x)}{x} = 0 \qquad \textit{(Density of primes)}[7],$$

which means that the number of primes is extremely small in comparison with that of real numbers.

$$\lim_{n\to\infty} \inf \frac{p_{n+1} - p_n}{\ln p_n} = 0 \qquad \textit{(Goldston, Pints Yildirin theorem on prime gaps)}[7],$$

revealing that the least possible prime gap can be exceedingly small.

$$\lim_{n\to\infty} \sup \frac{p_{n+1} - p_n}{\ln p_n} = \infty \qquad \textit{(Westzynthins maximal theorem on prime gaps)}[7],$$

which holds that the highest possible prime gap can be exceedingly large.

$$\lim_{n\to\infty} \inf p_{n+1} - p_n < 246 \qquad \textit{(Zhang, Tao, Maynard)}[6,\ 8,\ 10]$$

this establishes that the pair primes with a gap not less than 246 appear infinitely often.

$$n^{p-1} \equiv 1 \mod p \qquad \textit{(Fermat's litle theorem / primality test)}[2],$$

which implies the possibility of making use of congruent properties to quickly determine whether a positive integer is a prime or not without resorting to the ancient and crude trial division approach.

$$\{p, p+k, p+2k, ...p+mk\} \subset \{a, a+d, ..., a+rd\} \quad \textit{(Drichilet's theorem on primes in an AP)}[1,\ 2]$$

with the interpretation that prime numbers also exist in arithmetical progression.

$$p = \lfloor 2^{\cdot^{\cdot^{\cdot^{2^{2^{\mu}}}}}} : \ \mu > 0 \rfloor \qquad \textit{(Mills prime-generating formula)}[2],$$

showing that primes can be tested to exist with a formular.

$$n = p_i + p_j + p_k \qquad \textit{(Vinogradov's theorem)}[7]$$

meaning that every sufficiently large integer is a sum of three primes.

The above interesting list also includes a number of verified conjectures pertaining to primes that still remain open to this day. The like of twin prime conjecture, Goldbach conjecture, Polignac conjecture, Prime k-tuple conjecture are yet to be laid to rest, for there is still a lot ground to break concerning the mystery that surrounds the distribution of primes (see [2, 3, 5]). However, the goal of this paper, motivated by the concise and yet remarkable postulate of Bertrand which posits that a form for Euclid theorem of infinitude of prime is such that $n \leq p \leq 2n$ for all integer $n > 1$, is to seek the prevalence of pair primes $p, p+k$ in the interval $[n!, n+k!]$. This is fully achieved in the next session.

1.1 A postulate and general conjectures concerning pair primes from numerical investigation

We begin by putting forward the following trivial question saying: how many twin primes (pair primes with a gap of 2) are there between 2 and 24, cousin primes (pair primes with a gap of 4) between 2 and 720, and sexy primes (pair primes with a gap of 6) between 2 and 40320?

Of course, within a twinkle of an eye, the answer would be somewhat like at least 4, for each of those kind of pair primes; or relatively, the answer may sound like, at least 2, 4 and 6 which represent the value of the prime gap for twin primes, cousin primes and sexy primes respectively.

Here, it seems there is a pattern already such as i.e there are at least k pair primes of gap k between $n!$ and $(n+k)!$, but in order to avoid a hasty conclusion, we rather perform some computations (specifically for integer n such that $1 < n \leq 10$ and even k such that $1 < k \leq 10$) as exemplified in the tables below.

Note that for the rest of this paper $\lceil x \rceil$ *is used to represent the ceiling function which is the nearest* *integer not less than x i.e for example:* $\lceil 2.1 \rceil = 3$*;* $n!$ *represent the factorial function of* n*, for example:* $5! = 5 \times 4 \times 3 \times 2 \times 1$*; and* $\#(X)$ *represent the cardinality of the finite set* X*: for example* $\#\{(11,13),\ (17,19)\} = 2$*, since it contains two twin primes.* Also, we use $\mathbb{Z}_2$ to represent the set of integers above 2, $\mathbb{N}$ is taken as the set of natural numbers while the usual $|.|$ represents the modulus function. Then we define $\phi_{k,n} := \lceil \frac{\sqrt[k]{n}}{|n-k|+1} \rceil$ with which we perform the following computations:

Table 1.

n	k	$n!$	$(n+k)!$	$(p,\ p+k) : n! \leq p < p+k \leq (n+k)! : p,\ p+k \in \mathbb{P}$	$\phi_{k,n}$
2	2	2	24	$(3,5),\ (5,7),\ (11,13),\ (17,19)$	2
3	2	6	120	$(11,13),\ (17,19),\ (29,31),\ (41,43), ...$	1
4	2	24	720	$(29,31),(41,43),(59,61),(71,73), ...$	1
5	2	120	5040	$(137,139),(149,151),(179,181), ...$	1
6	2	720	40320	$(809,811),(821,823),(827,829), ...$	1
7	2	5040	362880	$(5099,5101),(5231,5233),(5279,5281), ...$	1
8	2	40320	3628800	$(40427,40429),\quad (40529,40531), ...$	1
9	2	362880	39916800	$(362951,362953),(363017,363019), ...$	1
10	2	3628800	479001600	$(3628967,3628969),(3629027,3629029), ...$	1

Table 2.

n	k	$n!$	$(n+k)!$	$(p,p+k) : n! \leq p < p+k \leq (n+k)!, p, p+k \in \mathbb{P}$	$\phi_{k,n}$
2	4	2	720	$(3,7),(7,11),(13,17),(19,23), ...$	1
3	4	6	5040	$(7,11),(11,17),(19,23),(37,41), ...$	1
4	4	24	40320	$(37,41),(43,47),(67,71),(79,83), ...$	2
5	4	120	362880	$(127,131),(163,167),(193,197),(223,227), ...$	1
6	4	720	3628800	$(739,743),(757,761),(769,773),(823,827), ...$	1
7	4	5040	39916800	$(5077,5081),(5161,5171),(5227,5231), ...$	1
8	4	40320	479001600	$(40357,540361),(40423,40427),(40429,40433), ...$	1
9	4	362880	6227020800	$(362983,362987),(363043,363047), ...$	1
10	4	3628800	87178291200	$(3628987,3628991),(3629107,3629111), ...$	1

Table 3.

n	k	$n!$	$(n+k)!$	$(p,p+k) : n! \leq p < p+k \leq (n+k)!, p, p+k \in \mathbb{P}$	$\phi_{k,n}$
2	6	2	40320	$(5,11),(7,13),(11,17),(13,19), ...$	1
3	6	6	362880	$(7,13),(11,17),(13,19),(17,23), ...$	1
4	6	24	3628800	$(31,37),(37,43),(41,47),(47,53), ...$	1
5	6	120	39916800	$(131,137),(151,157),(157,163), ...$	1
6	6	720	479001600	$(727,733),(733,739),(751,757), ...$	2
7	6	5040	6227020800	$(5081,5087),(5101,5107),(5107,5113), ...$	1
8	6	40320	87178291200	$(40351,40357),(40423,40429), ...$	1
9	6	362880	$1.30767E+12$	$(362897,362903),(362977,362983), ...$	1
10	6	3628800	$2.09228E+13$	$(3628841,3628847),(3628847,3628853), ...$	1

Table 4.

n	k	$n!$	$(n+k)!$	$(p, p+k) : n! \leq p < p+k \leq (n+k)!, p, p+k \in \mathbb{P}$	$\phi_{k,n}$
2	8	2	3628800	$(5,13),(11,19),(23,31),(29,37),...$	1
3	8	6	39916800	$(11,19),(23,31),(29,37),(59,67),...$	1
4	8	24	479001600	$(29,37),(59,67),(71,79),(89,97),...$	1
5	8	120	6227020800	$(149,157),(449,457),(479,487),...$	1
6	8	720	87178291200	$(743,751),(761,769),(1031,1039),...$	1
7	8	5040	$1.30767E+12$	$(5399,5407),(5783,5791),(6599,6607),...$	1
8	8	40320	$2.09228E+13$	$(40343,40351),(41141,41149),...$	2
9	8	362880	$3.55687E+14$	$(362903,362911),(362969,362977),...$	1
10	8	3628800	$6.40237E+15$	$(3629099,3629107),(3629671,3629679),...$	1

Table 5.

n	k	$n!$	$(n+k)!$	$(p, p+k) : n! \leq p < p+k \leq (n+k)!, p, p+k \in \mathbb{P}$	$\phi_{k,n}$
2	10	2	479001600	$(13,23),(19,29),(37,47),(43,53),...$	1
3	10	6	6227020800	$(13,23),(19,29),(37,47),(43,53),...$	1
4	10	24	87178291200	$(37,47),(43,53),(127,137),...$	1
5	10	120	$1.30767E+12$	$(127,137),(139,149),(421,431),...$	1
6	10	720	$2.09228E+13$	$(1543,1553),(7069,7079),(7489,7499),...$	1
7	10	5040	$3.55687E+14$	$(7069,7079),(7489,7499),(40351,40361),...$	1
8	10	40320	$6.40237E+15$	$(40351,40361),(40519,40529),...$	1
9	10	362880	$1.21645E+17$	$\{362941,362951\},(363757,363767),...$	1
10	10	3628800	$2.4329E+18$	$(3630259,3630269),(3631741,3631751),...$	2

Then, naturally from the above numerical investigation come the following postulate and its consequential more general conjectures:

Postulate 1. $\#\{p : p,\ p+k \in \mathbb{P},\ n! \leq p < p+k \leq (n+k)!\ \forall\ n \in \mathbb{N}_{\geq 2},\ k \in 2\mathbb{Z}_{\geq 2}\} \geq 1.$

A relative of Polignac's conjecture:

Conjecture 2. $\#\{p : p,\ p+k \in \mathbb{P},\ n! \leq p < p+k \leq (n+k)!\ \forall\ n \in \mathbb{N}_{\geq 2},\ k \in 2\mathbb{Z}_{\geq 2}\} \geq \lceil \frac{\sqrt[k]{n}}{|n-k|+1} \rceil.$

Certain relatives of Dirichilet's Theorem on primes in an arithmetic progression:

Conjecture 3. $\#\{p : p,\ p+k_1,\ p+k_2 \in \mathbb{P},\ n! \leq p < p+k_1 < p+k_2 \leq (n+k_\star)!\ \forall\ n \in \mathbb{N}_{\geq 2},\ k \in 2\mathbb{Z}_{\geq 2}, k_\star = k_1 + k_2\} \geq \lceil \frac{\sqrt[k_\star]{n}}{|n-k_\star|+1} \rceil.$

Conjecture 4. $\#\{p : p,\ p+k_1,\ p+k_2,\ ...,\ p+k_r \in \mathbb{P},\ n! \leq p < p+k_1 < p+k_2 < ...\ p+k_r \leq (n+k_\star)!\ \forall\ n \in \mathbb{N}_{\geq 2},\ k \in 2\mathbb{Z}_{\geq 2}, k_\star = \sum_{i=1}^{r}\} \geq \lceil \frac{\sqrt[k_\star]{n}}{|n-k_\star|+1} \rceil.$

2 Conclusions

The conjectures and the postulate based on the numerical evidence given above are presented for future study. We would be interested in seeing the formal proofs that would eventually elevate them to status of theorems. And so, we wish the readers and aspiring researchers the effrontery required in proving or disproving them. Although, it seems that the present available tools needs refinement in attacking such problems, according to Granville ([4]). However, Euclid's theorem on infinitude of primes and the celebrated result of Green and Tao [1] that there are arbitrary long primes in arithmetic progression gave the assurance that the conjectures are likely true. In addition, if we let $\pi(x)$ count the number of primes up to x, the prime number theorem tells us that

$$\pi((n+k)!) - \pi(n!) \approx \frac{(n+k)!}{\ln(n+k)!} - \frac{(n)!}{\ln(n)!} \geq 2\lceil \frac{\sqrt[k]{n}}{|n-k|+1} \rceil$$

which heuristically support Postulate 1 and Conjecture 2. Also relating with twin-prime asymptotic conjecture that $\pi_2(x) = \int_2^t \frac{1}{\ln^2 t} dt$, where $\pi_2(x) = \{p : p, p+2 \in \mathbb{P}, p \leq x\}$, one can easily check out that

$$\pi_2((n+k)!) - \pi_2(n)! = \int_2^{(n+k)!} \frac{1}{\ln^2 t} - \int_2^{n!} \frac{1}{\ln^2 t} dt \geq \lceil \frac{\sqrt[k]{n}}{|n-k|+1} \rceil$$

which altogether heuristically support Postulate 1 and Conjecture 2 given in this paper. Indeed [9] is a good source of motivation for establishing the conjectures in this paper.

Acknowledgements

We would like to appreciate Dr Etuechere Gozie, of the Redeemer University, Nigeria for his kindness by availing us the opportunity to access some useful materials. Much thanks also due to Dr Dinesh Panthi and Dr. Adeyemo K. M. for helping us overcome certain intricacies with the required latex template during the final submission of this paper.

References

[1] Ben, G. and Tao, T., 2008, The Primes Contain Arbitrarily Long Arithmetic Progression, *Annals of Mathematics*, 167(2), 481 - 547.

[2] Crandall, R. and Pomerance, C., 2006, *Primes: Problems and Progress; Celebrated problems and curiosity*, A Computational Perspective second Edition, Springer Verlag, 1 - 21.

[3] Granville, A., 1996, Harald Cramer and the distribution of prime numbers, *Scandinavian Actuarial Journal*, 1, 12 - 28.

[4] Granville, A., 2015, Primes in interval of Bounded Lengths, *Bulletin of the AMS*, 52 (2), 171 - 222.

[5] Hardy, G. H., Wright, E. M., Heath-Brown, D. R. and Silverman, J. H., 2009, *An introduction to the theory of numbers*, 6th edition, Oxford: The Clarendon Press, pp 6.

[6] J. Maynard., 2015, Small gaps between primes, *Annals of Mathematics*, 181(1), 383 - 413.

[7] Olver, F. W. J. and Lozier, D. W., 2010, *Functions of number theory*, Handbook of Mathematical Functions, National Institute of Standards and Technology, 638 - 645.

[8] D. H. J Polymath., 2014, New equidistribution estimates of Zhang type, *Algebra* & *Number Theory*, 8(9), 2067 – 2199.

[9] Sándor, J, Minitrovic D. S., Crstici B., 2006, *Handbook of number theory I*, Kluwer Academic Publishers, 2, 227 - 240.

[10] Y. Zhang., 2014, Bounded gaps between primes, *Annals of Mathematics*, 179(3), 1121– 1174.