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Associated Prime Ideals in Commutative Algebra

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Abstract

This paper aims to explain how an abstract mathematical object of higher mathematics is learnt and solved the given problem from the pedagogical perspectives resting the explanation based upon the solution of higher mathematical problem in abstract algebra. The purpose is not only to disseminate the solution of the problem but to unfold the learning realities in course of pursuing the pure discovery learning path. The problem of the study was connected to the understandings of associated prime ideals in commutative algebra. Factual materials including books, lecture notes and research articles were used to explore the concepts on associated prime ideals. The concepts learned were summarized, conceptualized, and then analyzed together with established facts of algebraic concepts and developed examples to interpret the abstract concepts: the associated prime ideals. The associated prime ideal was explored and understood within the conceptions of primary decomposition, Noetherian rings, and modules, and then characterized some fundamental notions of it on an A-module M over the Noetherian ring A. The results reveal that repeated reading, writing, rigorous thinking and linking, and engaging on abstraction are the basic characteristics of content researcher in developing and constructing new insights, ideas, and knowledge in higher mathematics from pedagogic perspective.

Introduction

Learning abstraction in commutative algebra is an ongoing phenomenon for researchers in higher mathematics education. Commutative algebra is essentially the study of com- mutative rings (Atiyah & Macdonald, 1969; Gathmann, 2013), and attendant structures, especially the ideals and modules (Clark, 2015). It has wider application as tools in the domain of researching higher mathematics including the application in algebraic geometry and algebraic number theory (Gathmann, 2013, p.3). It is now becoming one of the foundation stones of studying algebraic geometry because it provides complete local tools for the subject (Atiyah & Macdonald, 1969, p.vii).

Clark (2015) explained that commutative algebra is a necessary and useful prerequisite for the study of other fields of mathematics in which we are interested. It includes varieties of abstract concepts including commutative rings, subrings, ideals, modules, algebras, homomorphisms, homological algebra etc. Among these concepts prime ideals are the basic building blocks (Atiyah & Macdonald, 1969) in studying and researching commutative algebra. The prime ideals in a commutative ring can also be used to study the several properties of modules over such rings. That is why this paper intends to explain how we can associate this important object (prime ideal) of commutative rings with different notions of abstract mathematical concepts in commutative algebra. Here, A is considered a non-zero commutative ring with identity, unless otherwise stated.

I prepared this paper while conducting my PhD research on the existence of associated prime ideals and exact zero-divisors in commutative algebra. Thus, studying associated prime ideals in commutative algebra was one of my basic activities in conducting my PhD re- search. In this paper, I explored some basic characteristics of an associated prime ideals and then explained how I learnt this abstract object of commutative algebra from pedagogical perspective within the framework of established learning principles as stated in result and discussion section.

Objectives

The objectives of this paper are:

- To explain the basic characterization explored on an associated prime ideal of an A-module M in commutative algebra.
- To explain how abstract mathematical objects of higher mathematics are learnt andsolved the given problem from the pedagogical perspectives resting the explanation based upon the solution of higher mathematical problem in abstract algebra.

Methodology

This paper was prepared from the start of my learning journey of my PhD in commutative algebra. My research was neither a field study conducted in social science, nor a laboratory experimental research as conducted in natural science. But my research belongs to formal science and thus I applied formal deductive techniques of mathematics to established facts in commutative algebra. Regarding to learning abstraction in higher mathematics, I believe on radical constructivism (Glasersfled, 1996) which views that learners personally construct their own understanding of each mathematical concepts by actively participating the activity on learned idea and validated these understanding with accepted views of mathematics by reflecting their own construction (Ernst, 1997, p.29). Understanding with these philosophical assumptions, I explored the basic ideas, concepts, examples, theorems and other mathematical facts related to associated prime ideals in commutative algebra by repeated reading and writing the factual materials like books, lecture notes, articles, and miscellaneous including (Atiyah & Macdonald, 1969; Hungerford, 1974; Dummit & Foote, 1999; Matsumura, 1986; Lady, 1998). The new concepts that I learned were combined with the already learned algebraic concepts to develop the basic conceptions of associated prime ideals in commutative algebra. These combined concepts were again studied, analyzed, summarized, conceptualized, and concretized as much as possible with the connection of examples to explore the basic characterization of an associated prime ideal. The deductive approach (Ernst, 2004) was used to justify the statements related to associated prime ideals in commutative algebra.

Results and Discussions

This section includes the results and discussions of the several facts of commutative algebra to understand and characterization of associated prime ideal within several headings. The section begins with the review of conceptions related to learning principles in mathematics.

Understanding the General Principles for Learning Mathematics

There is no universal learning principle in learning mathematics for all learners in all situations. Different learning theories developed in the span of time exhibit different principles for learning mathematics in school to higher education. The learning theories so far can be classified into three broad categories: Behaviorism, cognitivism, and constructivism. Behaviorism views learning as a change in behavior of an individual within stimulus-response framework which are externally observable (Yilmaz, 2011). According to Montilla (2019), this view of mathematical learning has been influential in the widespread use of rote and practice methods.

However, cognitivism views learning as a mental process where human mind works actively and measure the conceptual change such that the primary emphasis is given on how knowledge is acquired, processed, stored, retrieved, and activated during the different phases of learning process (Yilmaz, 2011, pp. 204 -205). This view of learning supports the use problem solving approach in learning mathematics. Using this approach in solving problems of mathematics increases knowledge and reasoning skills among learners where mental thinking has significant role (Lessani, Md.Yunus, Bakar, & Khameneh, 2016).

On the other hand, constructivism views learning is an internal progressive cognitive process through which an individual, confronted by a new environment or new knowledge, recurrently modifies his/her knowledge or learning scheme, thereby constructing new meaning (Belanger, 2011, p. 28). There are two types of constructivism: Social constructivism and radical constructivism. Social constructivism developed from the social cultural theory of Vygotsky (1978) which views learning mathematical concepts from social perspective and thus believes that mathematical objects and reality are socially constructed and co-constructed through social activities of individual. According to Ernst (1997), this theory has emphasized the social domain of knowledge which affects the developing individual in some crucial formative way that the person constructs his/her meanings in response to his or her experience in social contexts.

Likewise, radical constructivism believes that learners personally construct their own understanding of each mathematical concepts by actively participating the activity based on learned idea and validate these understanding with accepted views of mathematics by reflecting their own construction (Ernst, 1997, p.29). This view of learning was developed within two principles: Knowledge is actively constructed by the cognizing subject and the function of cognition is adaptive to the situation to the learner's experience (Glasersfled, 1996, p.2).

I believe in the theory of radical constructivism while constructing new knowledge in learning mathematics. Within these views of learning, I reviewed and analyzed the funda- mental concepts of rings and modules to develop the concepts of associated prime ideals in commutative algebra. The revision and analysis of the prerequisites are essential activities to learn new concepts in commutative algebra. As I experienced from the literature of learning theories reviewed above, the learner in higher mathematics can learn abstract concepts by using rigorous mental thinking through repeated reading, writing, and linking with learned concepts. The learning will occur while engaging regularly with the texts of established mathematical facts, making connection of learned ideas, constructing examples of abstract mathematical concepts, solving problems, and trying to find conjecturesand research gap. These activities are almost individual, but with the help and support of experts/teachers/mentor. The following paragraphs display how I learned the abstract algebraic concepts: the associated prime ideals in commutative algebra.

Prerequisites for Learning Associated Prime Ideals

Learning is a marvelous journey; it is a journey to the wonderful places our minds may wander; discovery and insights are the journey's joys and destinations (O'Brien, 2014, pp.23-24). I started my journey of learning associated prime ideals by reviewing the basic conceptions of commutative algebra. A ring is a nonempty set A with two operations $+$ (addition) and $\cdot$ (multiplication) defined in it such that $(A, +)$ is an abelian group, $(A, \cdot)$ is a semigroup and A holds distributive properties of multiplication over addition (Hungerford, 1974). If $a \cdot b = b \cdot a$, $\forall a, b \in A$, then A is said to be a commutative ring. If there exist an element $1 \in A$ such that $a \cdot 1 = a$, $\forall a \in A$, then A is called the commutative ring with the identity. Here, I consider the commutative ring A with identity. If $1 = 0$, then ring A has only one element 0 and this ring is called the trivial ring or zero ring.

For example, $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are infinite commutative rings with identity. Likewise, A non-empty subset S of the ring A is said to be a subring of A if$S$ itself is a ring with the identity. Clearly $\mathbb{Z}$ is the subring of both $\mathbb{R}$ and $\mathbb{C}$, and $\mathbb{R}$ is subring of $\mathbb{C}$. The property is a "subring of" is clearly transitive (Dummit & Foote, 1999, p.229). An ideal I of a ring A is a non-empty subset of A which is an additive subgroup and $AI \subseteq I$, i.e., for every $a \in A$ and for all $x \in I$ we have $ax \in I$ (Atiyah & Macdonald, 1969). Each element of the ring generates the ideals. Clearly, (0) and (1) = A, are the ideals of A, called the trivial ideals. The ideal which is not equal to (1) is calledthe proper ideal in ring A.

From these reviews of basic concepts of ideals and the subring in the commutative algebra written by Atiyah and Macdonald (1969), I found the contradiction on the concepts of ideals so far, I understood in general ring. So far, I understood that every ideal is a subring of the ring, but now it is not true because to become S a subring of the ring A, we must have 1 in S. So, in general case $3\mathbb{Z}$ is subring and ideal of the ring $\mathbb{Z}$ of integer, but itis only the ideal in commutative algebra because it does not contain the multiplicative identity.

Let I be an ideal of the ring A then the quotient group A/I is a ring under the binary operations defined as: $(a + I) + (b + I) = (a + b) + I$ and $(a + I) \cdot (b + I) = (ab + I)$ for all $a, b \in A$ (Dummit & Foote, 1999), called the quotient ring of A by I. The elements of A/I are the cosets of I in A (Atiyah & Macdonald, 1969). An ideal P in a ring A is said to be a prime if $P \neq (1)$ and if $x, y \in P \Rightarrow x \in P$ or $y \in P$ as defined in (Atiyah &Macdonald, 1969). In other words, the ideal P of A for which A/P is an integral domain is called a prime ideal (Matsumura, 1986). But in (Hungerford, 1974), an ideal P in a ring A is said to be prime if $P \neq A$ and for any ideals I, J in A such that $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. These definitions are equivalent to each other. In Z, every ideal of the form (p) for some prime number p and the zero ideal (0) are prime ideals.

An ideal which is maximal among all proper ideals is called a maximal ideal or in other words the ideal m of A is maximal if and only if A/m is a field.

Let A be a commutative ring with identity. An additive abelian group M is said to be an A-module if there exist a function μ: $A \times M \to M$ given by $\mu\ (a, x) \rightarrowtail ax$ and satisfying (Atiyah & Macdonald, 1969): for all $a, b \in A, x, y \in M$ (i) $a\ (x + y) = ax + ay$

(ii) $(a + b)\ x = ax + bx$ (iii) $a(bx) = (ab)\ x$ and (iv) $1 \cdot x = x$.

If A is a field, then M is a vector space over field A. Every additive abelian group G is a $\mathbb{Z}$-module. Every ring A is an A-module. Note that if the ring has unity $1 \neq 0$ then the A-module is called unitary. The ideal I, quotient ring A/I, polynomial ring $A[x]$, ring of formal power series $A[[x]]$ etc. are example of A-modules. The submodule N of an A-module M is a subgroup of M which is closed under the multiplication of elements of A i.e., $ax \in N, \forall a \in A$ and $x \in N$ (Dummit & Foote, 1999, p.319). Let X be a subset of an A-module M, then the intersection of all submodules of M which contains X is a submodule generated by X, denoted by $\langle X \rangle$. If X is finite and N is the module generated by X, then N is said to be finitely generated A-module (Hungerford, 1974). If M is finitely generated A-module then M is simply called finite A-module or is finite over A (Matsumura, 1986, p.7).

The revision, analysis and understanding of these vary fundamental concepts are the basis for learning abstraction related to associated prime ideals in commutative algebra.

Primary Ideals and Associated Prime

Within these reviews of fundamental concepts of commutative algebra, I came to defineassociated prime ideals in relation to the concepts of primary and prime ideals.

A proper ideal I of a ring A is a primary ideal if $x\ y \in I \Rightarrow$ either $x \in I$ or $y^n \in I$ for some $n \geq 1$ (Atiyah & Macdonald, 1969, p.50). Equivalently, I is primary if and only if$A/I \neq 0$ and every zero divisor in A/I is nilpotent. For example, every prime ideal P is primary ideal, but the converse may not be true. In the ring of integer Z, the zero ideal (0), and the ideal (p^n), for some prime number p are primary ideals.

Theorem: *Let I be a primary ideal in a ring A, then rad (I) is the smallest prime idealcontaining I* (Atiyah & Macdonald, 1969, p.50).

If I is a primary ideal of a ring $P = r(I)$, then I is called the p-primary ideal. This theorem motivated me to define the radical of an ideal in commutative algebra. Let I be an ideal in a commutative ring A with identity. Then the radical of I is denoted by $r\ (I)$ or $\sqrt{I}$ defined as $r\ (I) = \{x \in A|\ x^n \in I$ for some integer $n > 0\}$ (Atiyah & Macdonald, 1969, p.8). Likewise, I defined nilpotent and unit element in the commutative ring.

An element $x \in A$ is said to be a nilpotent element of the ring A if there exists an integer $n > 0$ such that $x^n = 0$. A ring having no non-zero nilpotent element is called reduced. The set of all nilpotent elements in ring A forms an ideal, called nilradical. An element x in ring A is said to be a unit element if there exists an element y in A such that $xy = 1$, i.e., x divides 1. Every nonzero element in the field is a unit.

Proposition: *The nilradical of A is the intersection of all prime ideals of A* (Atiyah & Macdonald, 1969, p.5).

The intersection of all the maximal ideals in the commutative ring A is called the Jacob- son radical of A, denoted by $J(A)$. Hence, we conclude that the radical of a zero ideal (0) is the nilradical of ring A. Likewise, we need to define the colon ideal or ideal quotient to understand the associated prime ideal in this section.

Let I and J be two ideals in the ring A, then the set $(I : J) = \{x \in A : xJ \subseteq I\}$ is an ideal of A called the colon ideal of A. For, $x \in A$, we have $(I : x) = \{y \in A : xy \in I\}$,and $(0 : x) = \{a \in A : ax = 0\}$ is called the annihilator of x in A, denoted by Ann (x).Thus, the set Ann (x) is an ideal in the ring A.

The following are the facts which are found in (Atiyah & Macdonald, 1969, p. 51).

If $r(Q)$ is the maximal ideal, then Q is primary ideal. In particular, the power of maximal ideal m is m- primary.

If Q_i $(1 \leq i \leq n)$ are P-primary ideals, then $Q = \bigcap_{i=1}^{n} Q_i$ is P-primary ideal.

Let Q be a P-primary ideal, x an element of A, then (i) if $x \in Q$ then $(Q : x) = (1)$, (ii) if $x \notin Q$, then $(Q : x)$ is a P-primary ideals and (iii) if $x \notin P$, then $(Q : x) = Q$.

Let I be an ideal of a ring A. Then, I is said to have primary decomposition if I can be expressed as the finite intersection of primary ideals in A. That is, $I = \cap^{n}_{i=1} I_i$ (Atiyah & Macdonald, 1969). This decomposition is called the minimal if $r(I_i)$ is all distinct and $I_i \not\supseteq \cap_{i \neq j} I_j$ ($1 \leq i \leq n$). The ideal which has primary decomposition is called the decomposable ideal.

For example, $16\mathbb{Z} = 2\mathbb{Z} \cap 4\mathbb{Z} \cap 8\mathbb{Z} \cap 16\mathbb{Z}$ is the primary decomposition which is not minimal primary decomposition. Here radical of each primary component is $2\mathbb{Z}$. But $30\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z}$ is the minimal primary decomposition where the radical of each component is distinct and none of which component contains the intersection of the remaining components. These examples show that the primary decomposition of the ideals in thering $\mathbb{Z}$ of integer is related to the prime factorization of the integers because $30 = 2\times3\times5$ and $16 = 2^4$. Thus, we can find the primary decomposition of every ideal in the ring $\mathbb{Z}$ of integers.

The following uniqueness theorem (Atiyah & Macdonald, 1969, p.52) is the basis for the definition of associated prime ideals belonging to the ideal I.

(First Uniqueness Theorem): *Let I be a decomposable ideal and let $I = \cap_{i=1}^{n} I_i$ be a minimal primary decomposition of I. Let $P_i = r(I_i)$ ($1 \leq i \leq n$). Then, the P_i are precisely the prime ideals which occur in the set of ideals $r(I : x)$ ($x \in A$) and hence independent of the particular decomposition of I.* (Here x varies on all the elements of A).

Thus, we obtain the set of prime ideals from the minimal primary decomposition of the decomposable ideal I. These set of prime ideals are called the prime ideals belonging to Ior the AM-associated prime ideal of I. Here AM means Atiyah and McDonald definition of associated prime ideals of I. Now, we can define the AM-associated prime ideals as follows.

Let $I = \cap^{n}_{i=1} I_i$ be the minimal primary decomposition of I and $r(I_i) = P_i$. Then, the set $\{P_1, P_2, \ldots, P_n\}$ is called the set of AM-associated prime ideals belonging to I. The set of all AM-associated prime ideals of I *is* denoted by $Ass^{AM}(I)$. For example, the ideal (12) in $\mathbb{Z}$ has the minimal primary decomposition $(12) = (2^2) \cap (3)$. Thus, the set $\{r(2^2), r(3)\} = \{(2), (3)\} = \text{Ass}(12)$. Hence, (2) and (3) are the prime ideals associated to (12) in the ring $\mathbb{Z}$ of integers.

If we consider A/I is an A-module, then the associated prime ideals given by the uniqueness theorem are those which occurs the radical of the annihilator of the element of A/I.To prove this, let P be an associated prime ideal of I. Then $P = r(I : x)$ for some $x \in A$, $x \notin I$ because $x \notin I_i$ for some i, but $x \in \cap_{i \neq j} I_j$. Let $a \in P$ then $a^n \in (I : x)$ for some $n > 0$. This implies that $a^n x \in I$ and $a^n x + I = I$. This implies that $a^n \in (\bar{0} : \bar{x})$ and $a \in r(\text{Ann}((\bar{x}))$. So, $P \subseteq r(\text{Ann}((\bar{x}))$, for some nonzero $\bar{x} \in A/I$. Conversely, assume that $a \in r(\text{Ann}((\bar{x}))$, for some nonzero $\bar{x} \in A/I$. Then, $a^n x + I = I$ which implies $a^n \in (I : x)$ and $a \in r(I : x) = P$. Thus, $\text{Ann}((\bar{x}) \subseteq P$ implies that $P = r(\text{Ann}((\bar{x}))$, for some nonzero $\bar{x} \in A/I$.

In this sense, the prime ideals associated to the ideal I is exactly same with the prime ideal associated with the A-module A/I. This justification gave an insight that the prime ideal associated with some decomposable ideal can also be computed with the help of module. Among the set of AM-associated prime ideals, the minimal elements are called the minimal or isolated prime ideals, and the remaining are the embedded prime ideals.

Clearly, an ideal I is primary if and only if it has only one AM- associated prime ideal, namely its radical (Sosna, n.d.). This means that the primary ideal has no embedded associated prime ideals.

For example, let $A = k[x, y]$ and $I = (x^2, xy)$. Then $I = P_1 \cap P^2$ where $P_1 = (x)$ and $P^2 = (x^2, xy, y^2)$. Since $r(P_2)^2 = P_2 = (x, y)$ a maximal ideal in A. Thus, $P_2^{\,2}$ is primary, and P_1 is a prime ideal, so it is primary. Thus, I has primary decomposition with AM- associated prime ideals P_1 and P_2. Here, P_1 is minimal associated prime and P_2 is the embedded associated prime ideals.

Thus, the linking with learned ideas are essential activities to understand the new abstract concepts, particularly, the associated prime ideals in commutative algebra. Without understanding the concepts of primary ideals, colon ideals, annihilator ideals, primary decomposition and related relations, it is very difficult to grasp the concepts of AM-associated prime ideals. So, rigorous mental activity is essential to conceptualize abstraction by thinking and linking with established facts. Also, the examples can help to conceptualize abstraction. If the ring is Noetherian then we can visualize associated prime ideals differently as described in the following section.

Learning Associated Prime Ideals in Noetherian Rings

A ring A is called a Noetherian ring if it satisfies the ascending chain condition of ideals. This means that for any increasing sequence of ideals $I_1 \subseteq I_2 \subseteq \ldots$ there exist a positive integer n such that $I_n = I_{n+1} = I_n$ other words, if every ideal of the ring A is finitely generated (Atiyah & Macdonald, 1969). Similarly, an A-module M *is* said to be a Noetherian module if it satisfies the ascending chain condition on its submodules. Equivalently, if every submodule of A-module M *is* finitely generated, then M *is* called a Noetherian A-module. The following examples are constructed while reviewing the literature (Atiyah & Macdonald, 1969; Matsumura, 1986; Dummit & Foote, 1999).

The finite rings, field, and the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$ are some examples of Noetherian rings. If A is a Noetherian ring and I is an ideal of A, then the quotient ring A/I is also Noetherian. The rings $k[x]$, $k[[x]]$, $k[x_1, \ldots, x_n]$ and $k[[x_1, \ldots, x_n]]$ are all Noetherian where k is any field but the ring $k[x_1, \ldots, x_n]$ of infinite number of variables is not Noetherian ring. This is the subring of the quotient field $F = k(x_1, x_2 \ldots,)$ where F is a Noetherian ring. This shows that every subring of the Noetherian ring may not be Noetherian.

An ideal I of a ring A is said to be an irreducible ideal if $I = J \cap K \Rightarrow I = J$ or $I = K$ (Atiyah & Macdonald, 1969, p.82). Otherwise, the ideal I is called reducible. In the ring $\mathbb{Z}_6$, the ideal (2) is irreducible, while the ideal (6) is reducible in $\mathbb{Z}$. In a Noetherian ring A, every ideal is the finite intersection of irreducible ideals, every irreducible ideal is primary, every ideal has primary decomposition, every ideal I contain a power of its radical and the nilradical is nilpotent. Within this understanding of Noetherian ring, we can characterize the associated prime ideals.

Proposition: *Let* I *be a proper ideal in a Noetherian ring. Then the prime ideals which belongs to* I *are precisely the prime ideals which occur in the set* $(I : x)$, *where* x *varies in* A.

This proposition gives that the concepts of associated prime ideals is more general than as given by uniqueness theorem in general ring. That is the prime ideals associated to the ideals I or an A-module A/I are those prime P of A such that $P = \text{Ann}(x)$, for some non-zero x in A/I. If $I = 0$, then the prime ideal associated to I is just the prime ideal associated to the ring A.

According to Matsumura, see in (Matsumura, 1986, pp. 38 - 42), for a commutative ring A and an A-module M, a prime ideal P of A is called an associated prime of M if P is the annihilator $Ann(x)$ of some $x \in M$ (Matsumura, 1986, p.38). The set of all associated prime of M is denoted by $Ass(M)$ or $Ass_A(M)$. This definition is also understood as the modern notion of associated prime ideals. The earlier definition of AM-associated prime ideal is exactly same as this modern notion if the given commutative ring is Noetherian. If I is an ideal of a ring A then A/I is an A-module, and thus the associated prime ofA/I are the prime ideal of A which are the prime divisors of I (Matsumura, 1986). For example, in $\mathbb{Z}$, the prime ideal associated to $24\mathbb{Z}$ are $2\mathbb{Z}$ and $3\mathbb{Z}$ which are the prime divisors of $24\mathbb{Z}$.

If the ring is not Noetherian, then these two definitions of the conception of associated prime ideals are different. For example, if $A = k[x_1, x_2, \ldots]$ is the polynomial ring of countably infinite variables x_i, then $I = (x_1^2, x_2^2, \ldots)$ is not a prime ideal in A which is the annihilator of $1 \in (A/I)$. But its radical $P = (x_1, x_2, \ldots)$ is a prime ideal which is not annihilator of any element of A/I (Savitt, 2000, p.1). Furthermore, a prime ideal P is associated prime for M if and only if M contains a submodule isomorphic to A/P, see in (Lady, 1998). For, $m \in M$, if $P = Ann(m)$ then A/P is isomorphic to the cyclic submodule Am (that is the submodule generated by single element).

The above discussions on the conception of associated prime ideals show that understanding it is not an easy task. These concepts are abstract in nature. Abstract algebra requires thinking with high level abstraction and it explores the possible relationship among abstraction (Manandhar & Sharma, 2021). Thinking and linking with the learned concepts is essential to conceptualize such abstract definitions. The examples are only the visualization and concretization of such abstraction in commutative algebra. But Pinter (2010) explained that abstraction is all relative; one person abstraction is anotherperson's bread and butter, and so the abstract tendency in mathematics is a little like the situation of changing moral codes (p.12). It means that the connection between different examples, relations and learned facts with new concepts can make easy conceptualization for abstraction.

Learning basic Characterization of Associated Primes

After conceptualizing associated prime ideals, I explored some characteristics of it and tried to prove it in my own language in the following theorems. These theorems are not the original constructions of myself, but I tried to give proof in an easy manner as much as possible. I considered A to be a non-zero commutative Noetherian ring with identity and M is a finitely generated A-module here after.

Theorem: *The prime ideal P is an associated prime ideal of an A-module M if and only if there exists an injective map from A/P to M. If N is submodule of M, then $Ass(N) \subseteq Ass(M)$.*

Proof: Since P is the annihilators of $0 \neq xM$ then define a function $f : A/P \to M$ by $f(a + P) = ax$ then f is an A-module homomorphism because $f(k(a + P) + (b + P)) = f((ka + b) + P) = (ka + b)x = kax + bx = kf(a + P) + f(b + P)$. To show f is injective, let $(a + P) \in \operatorname{Ker} f$, then $f(a + P) = ax = 0$. This implies that $a \in \operatorname{Ann}(x) = P$. Thus $a + P = P$ shows $\operatorname{Ker} f$ contains only the zero element of A/P.

Conversely, assume that f is injective and assume $f(1 + P) = x \neq 0$ lies in M. Now it remains to show that $P = \operatorname{Ann}(x)$. For, let $a \in P$ then $a + P = P$ and $f(a + P) = f(P) = f(0 + P) = 0.x = 0$. Then $ax = 0$ implies $a \in \operatorname{Ann}(x)$. On the other hand, if $a \in \operatorname{Ann}(x)$, then $ax = 0$ implies $f(a + P) = 0 = f(0 + P)$ and by injectivity of f, we have $a + P = P$ and so $a \in P$. Hence, $P = \operatorname{Ann}(x)$ with non-zero $x \in M$ and so P is an associated prime of M.

Moreover, if $P \in \operatorname{Ass}(N)$ then $P = \operatorname{Ann}(x)$ with $0 \neq x \in N \subseteq M$ implies $x \in M$. Thus

$P \in \operatorname{Ass}(M)$.

Theorem: *Let A be a Noetherian ring and M be an A-module. Let $F = \{\operatorname{Ann}(x): 0 \neq x \in M\}$, then the maximal element in F is an associated prime of M.*

Proof: Let $P = \operatorname{Ann}(x)$, the maximal element in F and $ab \in P$. If $a \notin P$, then $ax \neq 0$ in M. Clearly, $P = \operatorname{Ann}(x) \subseteq \operatorname{Ann}(ax)$ and $b \in \operatorname{Ann}(ax)$. But P is maximal in the set F implies $\operatorname{Ann}(ax) \subseteq P$. Thus $P = \operatorname{Ann}(ax)$ and so $b \in P$ implies P is an associated prime of M.

Theorem: *Let M is a module over a Noetherian ring A, then $Ass(M)$ is nonempty if and only if $M \neq 0$.*

Proof: Assume $M \neq 0$ then the maximal element in the set F of annihilators ideals of nonzero elements of M is an associated prime of M shows $\operatorname{Ass}(M)$ is non-empty set.

Conversely, assume that $\operatorname{Ass}(M)$ is non-empty set. Then there exist at least one P in $\operatorname{Ass}(M)$ such that $P = \operatorname{Ann}(x)$ for $0 \neq x \in M$. Thus, M is a non-zero A-module.

Definition: Suppose M be an A-module. An element $a \in A$ is called a zero divisor for M if there is a non-zero $x \in M$ such that $ax = 0$, otherwise that a is M-regular (Matsumura, 1986, p.38).

Theorem: *The set $Z(M)$ of all zero divisors of M is equal to the union of associated prime ideals of A-module M.*

Proof: Let $a \in \cup_{P \in Ass(M)} P$. Then $a \in P$ *for* some $P \in \operatorname{Ass}(M)$. This implies that $P =$

$\operatorname{Ann}(x)$ for $0 \neq x \in M$. That is $ax = 0$ implies $a \in Z(M)$. Thus, $\cup_{P \in Ass(M)} P \subseteq Z(M)$ Conversely, assume a is a zero divisor of M. Then $ax = 0$ for non-zero $x \in$

M. This implies that $a \in$ Ann (x) and thus a belongs to the maximal element of the collection of such annihilator ideals which is an associated prime of M because A is a Noetherian ring. Thus $a \in \cup_{P \in Ass(M)} P$. Hence, we get $Z(M) = \cup_{P \in Ass(M)} P$.

Theorem: *For every prime ideal P, Ass* $(A/P) = \{P\}$.

Proof: Since the identity map ι: $A/P \to A/P$ is an A-module monomorphism, then clearly P is an associated prime ideal of A/P. If Q is also an associate prime ideal of A/P then Q = Ann $(x + P)$ where $x \notin P$. Then $y \in Q \iff xy + P = P \iff xy \in P \iff y \in P\,(x \notin P) \iff P = Q$.

These explanations and proofs of some characteristics of associated prime ideals of the modules in Noetherian commutative rings show that the abstract mathematical concepts are generally justified by using deductive method of proving. The relations of such abstraction are always linked with the established facts. The learner not only derives the proof of the relation of abstraction, but also learns how and why these concepts can learn in a sequence as a tool to extend new knowledge in commutative algebra.

Conclusion

Associated prime ideal is a fundamental concept for studying various properties of commutative ring and modules in commutative algebra. There are several notions of fundamental algebraic concepts required to conceptualize the abstraction related to associated prime ideals. The rigorous mental thinking and linking with the prerequisite, including the concepts of primary decomposition; prime ideals; primary ideals; annihilators; zero divisors; colon ideals; radical of ideals and the Noetherian property of rings and modules, are necessary activities for learning such abstraction in commutative algebra. Constructing examples of such abstract concepts is the basic skills required for meaningful understanding among the learner and researcher in it. As described by radical constructivism, the learner can construct knowledge, ideas and develop insights related to abstract concepts in commutative algebra while learning and doing research in it. There are different ideas of associating prime ideals of commutative rings including the techniques of primary decomposition. The AM-associated prime ideals and associated prime ideals of an A-module M are same where A is a Noetherian ring. Thus, a prime ideal P is said to be an associated prime ideal of an A-module M if it is the annihilatorof some non-zero element in M. In a Noetherian ring, it is always guaranteed that the existence of associated prime ideals of a non-zero A -module M, and the union of all such associated prime ideals of M is equal to the set of all zero divisor of M.

Rigorous mental thinking, construction of examples and developing linking chain insightswith basic notions of abstract algebraic concepts are essential to conceptualize abstraction in higher mathematics. Engaging with reading and rereading the factual materials like books, lecture notes and research articles together with thinking and linking with learned concepts are the essential pedagogical ways of constructing new knowledge, ideas, and insights in the journey of learning abstraction in higher mathematics. The learner can learn such abstract concepts like associated prime ideals in commutative algebra within the notion of radical constructivism while studying in higher mathematics.

Thus, this research helps the content researcher in higher mathematics in commutative algebra to understand the basic concepts and characteristics of an associated prime ide al. Equally, it is important from pedagogic perspective because it explored that repeated reading, mental thinking, and linking with learned facts are fundamental activities of the content researcher while constructing new knowledge in higher mathematics.

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