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The Performance of CIP Scheme in Solving Advection Equation

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Abstract

Constrained interpolation profile (CIP) approach is reviewed in this paper. This research provides a performance comparison between CIP and upwind scheme, a different general method for solving advection equations. The performance of numerical method will be judged based on its accuracy in solving advection equation. The simulation results indicate that CIP scheme advection is more accurate than Upwind scheme advection due to its minor numerical deviation from exact solution.

Keywords: Constrained Interpolation Profile, Upwind scheme, Numerical Diffusion, Advection

1.0 Introduction

Computational Fluid Dynamics (CFD) has played an important role in scientific research, influencing fields such as civil engineering, aerospace engineering, oil and gas industries and etc. Fluid simulation is one of the cores of CFD, it holds significant importance across various scientific, engineering, and entertainment applications due to its ability to model and predict the behavior of fluids in a controlled virtual environment. Advection equation is a partial differential equation that is fundamental to many computational models of fluid simulations because it governs the movement of scalar quantities in fluid flows. In the pursuit of accurate and stable numerical solutions to that equation, many numerical schemes such as Upwind, Monotone Upstream-centered Schemes for Conservation Laws (MUSCL), Constrained Interpolation Profile (CIP), have emerged as methods of the particular interest.

This paper presents an evaluation of CIP scheme performance in solving advection equations. The CIP scheme performance will be compared to Upwind Scheme performance. Our research focuses on evaluating the accuracy of Upwind and CIP scheme in advection dominated problems.

2.0 Governing Equation

The governing equation used for this study is called one-dimensional advection equation. The one-dimensional form of the advection equation is given by:

$$\frac{\partial \emptyset}{\partial t} + u\frac{\partial \emptyset}{\partial x} = 0 \qquad \text{(Equation 1)}$$

Where, ϕ is the quantity being advected (e.g., water level, temperature), t is time, x is the spatial coordinate, and u is the constant advection velocity.

This equation describes the transport of the quantity ϕ by a constant velocity field, u. The term $\frac{\partial \emptyset}{\partial t}$ represents the rate of change of ϕ with respect to time, and $\frac{\partial \emptyset}{\partial x}$ represents the advective transport term along the spatial coordinate x.

This advection equation is a first-order partial differential equation. It is a fundamental equation in fluid dynamics and is often encountered in various scientific and engineering applications where the transport of quantities by a flowing fluid needs to be modeled.

3.0 Literature Review

3.1 Constrained Interpolation Profile (CIP)

The Constrained Interpolation Profile (CIP) is a numerical technique used in computational fluid dynamics (CFD) and related fields for solving partial differential equations (PDEs) that describe the behavior of fluid flows. CIP is particularly well-suited for problems dominated by advection, where the transport of fluid properties plays a crucial role. The key strength of CIP lies in its capacity to provide accurate and stable solutions for fluid dynamics simulations, especially in situations characterized by advection and complex flow phenomena.

CIP scheme is a semi-Lagrangian scheme which traces and advects discretized derivatives and information through the fixed grid and control volume. CIP scheme uses spatial derivatives and spatial gradients to reconstruct a solution that is close to the real solution using cubic polynomial interpolation profile (Yabe et al, 2001). This scheme not only has third order accuracy in time but also space derivatives (Yabe et al., 2004). Besides, CIP scheme is stable, less diffusive and only contain small error in numerical simulation (Utsumi et al., 1996). This scheme has been proven by many researchers that it is a superior and reliable scheme.

Besides, CIP scheme is actually a method that reconstruct cubic polynomial function as shown in equation 2 to predict advection quantity and gradient in the next time step. Substitute and differenctiate the function with x_i and x_{i-1} to get equation 3, 4, 5 and 6. Formulate a_i and b_i as equation 7 and 8 after solving equation 3, 4, 5 and 6 simulataneously. Define $\xi = -u\Delta t$, function quantity and gradient in next time step, ${f_i}^{n+1}$ and ${g_i}^{n+1}$ can be predicted and advected as equations 9 and 10. $D = -\Delta x, iup = i - 1$, for $u \geq 0$; and $D = \Delta x, iup = i + 1$, for $u < 0$ (Yabe et al., 2001)

$$F_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \quad \text{(Equation 2)}$$

$$F_i(x_i) = d_i = f_i \quad \text{(Equation 3)}$$

$$\frac{dF_i(x_i)}{dx} = c_i = g_i \quad \text{(Equation 4)}$$

$$F_i(x_{i-1}) = -a_i\Delta x^3 + b_i\Delta x^2 - c_i\Delta x + d_i = f_{i-1} \quad \text{(Equation 5)}$$

$$\frac{dF_i(x_{i-1})}{dx} = 3a_i\Delta x^2 - 2b_i\Delta x + c_i = g_{i-1} \quad \text{(Equation 6)}$$

$$a_i = \frac{2(f_i - f_{iup})}{D^3} + \frac{g_i + g_{iup}}{D^2} \quad \text{(Equation 7)}$$

$$b_i = \frac{3(f_{iup} - f_i)}{D^2} - \frac{2g_i + g_{iup}}{D} \quad \text{(Equation 8)}$$

$${f_i}^{n+1} = a_i\xi^3 + b_i\xi^2 + {g_i}^n\xi + {f_i}^n \quad \text{(Equation 9)}$$

$${g_i}^{n+1} = 3a_i\xi^2 + 2b_i\xi + {g_i}^n \quad \text{(Equation 10)}$$

3.2 First-Order Upwind Scheme

First-order Upwind scheme is a fundamental method used in computational fluid dynamics (CFD) to solve advection problems. The fundamental idea of upwind scheme is to utilize the information from upstream to advect the flow quantity. The formula of flow quantity is dependent on the flow direction or the sign of constant velocity, u of the flow. If constant velocity, u is positive, the left side of grid point, i is called upstream side while the right side is the downstream side, the traveling wave solution of the equation above propagates towards the right. Similarly, if constant velocity, u is negative, the left side is called downstream side and right side is the upstream side, the traveling wave solution propagates towards the left.

Moreover, First-order Upwind scheme is a method to discretize equation 1, the governing equation into equation 11 or 12 depends on the sign of constant velocity, u then rearrange the equation and formulate ${\emptyset_i}^{n+1}$ as equation 13 or 14. The mechanism of Upwind Scheme is to predict and advect the flow quantity in next time step by using the ${\emptyset_i}^{n+1}$ formula. (H. Pletcher et al., 2013)

$$\frac{{\emptyset_i}^{n+1} - {\emptyset_i}^{n}}{\Delta t} + u\frac{{\emptyset_i}^{n} - {\emptyset_{i-1}}^{n}}{\Delta x} = 0, \text{for } u > 0 \qquad \text{(Equation 11)}$$

$$\frac{{\emptyset_i}^{n+1} - {\emptyset_i}^{n}}{\Delta t} + u\frac{{\emptyset_{i+1}}^{n} - {\emptyset_i}^{n}}{\Delta x} = 0, \text{for } u < 0 \qquad \text{(Equation 12)}$$

$${\emptyset_i}^{n+1} = -\frac{u\Delta t}{\Delta x}({\emptyset_i}^{n} - {\emptyset_{i-1}}^{n}) + {\emptyset_i}^{n}, \text{for } u > 0 \qquad \text{(Equation 13)}$$

$${\emptyset_i}^{n+1} = -\frac{u\Delta t}{\Delta x}({\emptyset_{i+1}}^{n} - {\emptyset_i}^{n}) + {\emptyset_i}^{n}, \text{for } u < 0 \qquad \text{(Equation 14)}$$

4.0 Methodology

In this study, numerical simulations are implemented by using Fortran to investigate the performance of Upwind scheme and CIP scheme. The computational domain was discretized with a grid spacing of Δx=1.0 and a total of 100 grid points. The initial condition and parameters of are illustrated in Tables 1 and 2 respectively.

The simulation results are visualized using Tecplot. This enables a comparative analysis of Upwind and CIP schemes. The performance of Upwind and CIP scheme will be compared to exact solution for accuracy evaluation. The duration of the simulations is 100 s. The initial condition for the Upwind scheme and CIP scheme is shown in Figures 1.

Table 1: Initial Condition of Upwind, CIP and Step Function

x (m)	*F*
$0 < x < 10$	0.0
$10 \leq x \leq 30$	10.0
$30 < x < 100$	0.0

Table 2: Parameters of Upwind and CIP Scheme Used to solve the Advection Equation

Parameter	Value
Velocity, u	0.5 m/s
Grid size, Δx	1.0 m
Time step, Δt	1.0 s

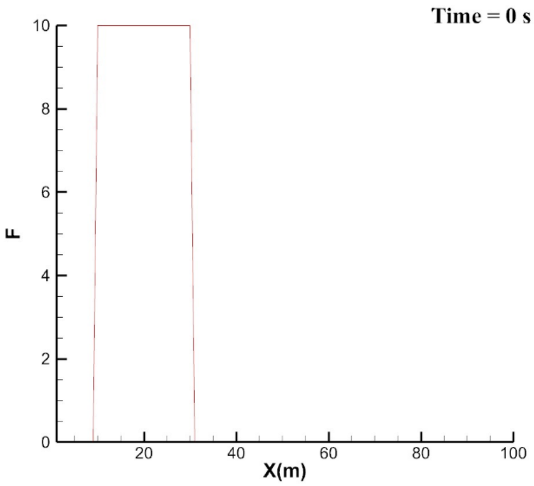

Figure 1: Initial condition for CIP and Upwind scheme

5.0 Results and Discussion

5.1 First-Order Upwind Scheme Advection

Figures 2, 3 and 4 reveal the characteristics of upwind scheme in the behavior of the transported quantity over time. As illustrated in the figures, significant change can be observed in the Upwind solution compared to exact solution. Initially, the Upwind solution accurately captures the leading edge of the exact solution and preserves the shape of the profile. However, as time progressed, a gradual deviation occurred in the upwind solution from the exact solution. The longer the time progresses, the more the deviation occurred in the Upwind solution.

The deviation of the Upwind solution from the exact solution indicates that the upwind scheme tends to diffuse the transported quantity. This behavior is consistent with the numerical characteristics of the Upwind scheme, where the numerical diffusion becomes more prominent over extended simulation durations. The observed discrepancy between the Upwind solution and the exact solution highlights the limitations of the upwind scheme in accurately preserving initial profile over the time periods. This diffusion effect emphasizes the need for consideration about the weakness of upwind scheme in the application of advection problems.

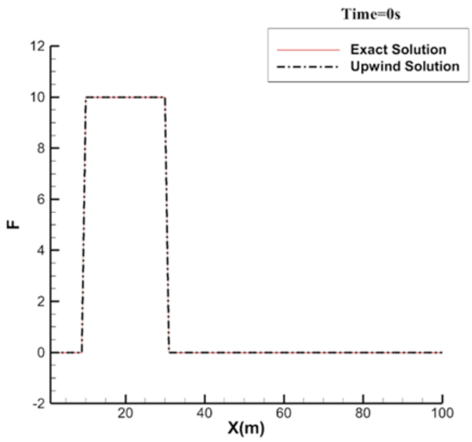

Figure 2: Advection of Upwind Solution and Upwind Solution at Time = 0 seconds

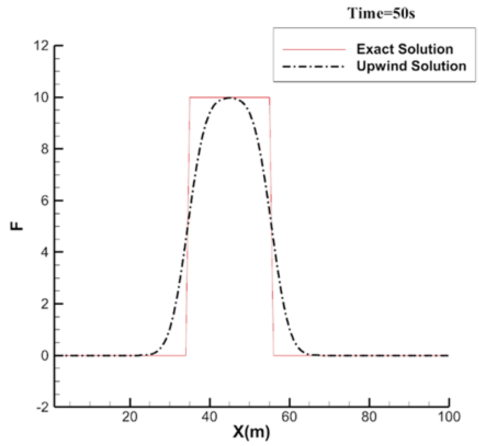

Figure 3: Advection of Upwind Solution and Exact Solution at Time = 50 seconds

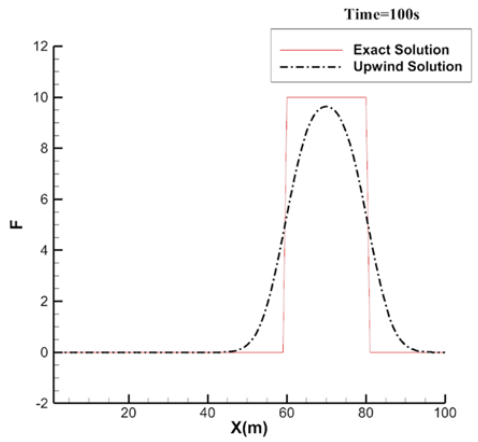

Figure 4: Advection of Upwind Solution and Exact Solution at Time = 100 seconds

5.2 CIP Scheme Advection

CIP scheme performance is exhibited in Figures 5, 6 and 7. Although there is a slight deviation between CIP solution and exact solution, the discrepancy is relatively minor in comparison to the performance demonstrated by the CIP scheme. This stability characteristic is a crucial attribute that guarantees the CIP solution remains accurate and reliable over the simulation period. The disadvantage of Upwind scheme becomes more obvious when it is compared to CIP scheme as the accuracy of Upwind solution decreases over time due to diffusion.

The superior performance of the CIP scheme highlights its effectiveness in solving advection equations with minimal numerical diffusion. The performance of CIP scheme is more accurate compared to Upwind scheme although there are slight variations from exact solution. The stability and accuracy exhibited by CIP scheme make it a desirable option for advection dominated problem which requiring precision in simulate the transport of quantities through fluid medium. The comparative analysis highlighted the constraints of upwind scheme and CIP scheme, emphasizing the advantageous characteristics of CIP scheme over upwind scheme for advection simulation.

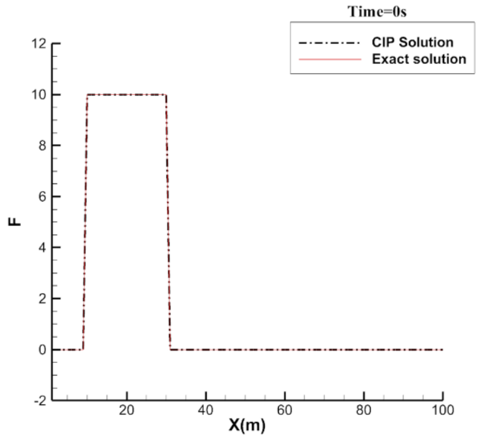

Figure 5: Advection of CIP Solution and Exact Solution at Time = 0 seconds

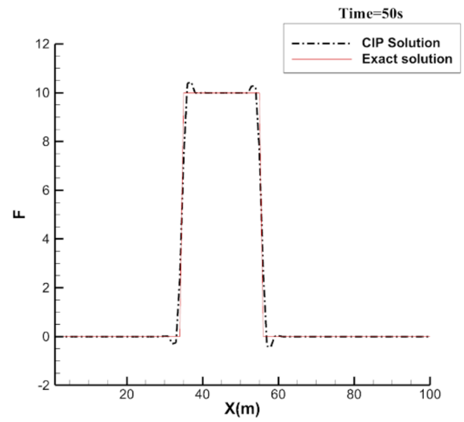

Figure 6: Advection of CIP Solution and Exact Solution at time = 50 seconds

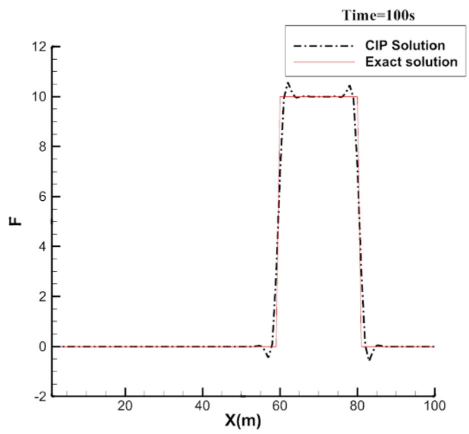

Figure 7: Advection of CIP Solution and Exact Solution at Time = 100 seconds

6.0 Conclusion

The comparative analysis of Constraints Interpolation Profile (CIP) and Upwind schemes in advection simulation offers valuable insights on their performance in accuracy. CIP scheme emerges as a stable and reliable numerical method by demonstrating minimal numerical diffusion and maintaining accuracy over time. Despite its minor deviation from the exact solution, the stability of CIP scheme is a significant advantage that enables CIP scheme suitable for simulations where accuracy and stability are important.

Meanwhile, the Upwind scheme exhibits significant numerical diffusion leading to substantial discrepancy from the exact solution. This diffusion effect highlights the limitations of Upwind Scheme in capturing transport quantity in advection equation. The Upwind scheme should be modified for higher order accuracy to avoid its disadvantageous characteristics.

In conclusion, CIP scheme proves to be a reliable and effective numerical method for advection simulations, providing superior performance in stability and accuracy. The findings of this study contribute to a better understanding of the characteristics of the numerical schemes and inform researchers in selecting the most appropriate method to achieve their specific simulation requirements.

7. References

H. Pletcher, R., C. Tannehill, J. & A. Anderson, D., 2013. *Computational Fluid Mechanics and Heat Transfer Third Edition.* Boca Raton: Taylor & Francis Group.

Li, Q., Omar, S., Deng, X. & Yokoi, K., 2017. Constrained Interpolation Profile Conservative Semi-Lagrangian Scheme Based on Third-Order Polynomial Functions and Essentially Non-Oscillatory (CIP-CSL3ENO) Scheme. *Communications in Computational Physics,* pp. 765-788.

Utsumi, T., Kunugi, T. & Aoki, T., 1996. Stability and accuracy of the Cubic Interpolated Propagation scheme. *Computer Physics Communications,* pp. 9-20.

Yabe, T. et al., 2004. Higher-order schemes with CIP method and adaptive Soroban grid towards mesh-free scheme. *Journal of Computational Physics,* pp. 57-77.

Yabe, T., Xiao, F. & Utsumi, T., 2001. The Constrained Interpolation Profile Method for Multiphase Analysis. *Journal of Computational Physics,* pp. 556-593.