ISSN 2091-1521 (Print)
2091-153X (Online)
DOI: https://doi.org/10.3126/ijorn.v11i1.64327
International Journal of Operational Research/Nepal - IJORN - 2023, Vol. 11, Issue 1

Golden Ratio: Construction, Geometry, Beauty, and Diversity

Iswar Mani Adhikari
Tribhuvan University, Prithvi Narayan Campus, Pokhara, Nepal
Email:ishwor.adhikari@prnc.tu.edu.np/adhikariim35@gmail.com

ABSTRACT

Nature is attractive due to the proper ratios of various components in them and in between others. In nature, leaves, flowers, fruits, Himalayas, hills, valleys, springs, rivers, etc. show a rhythmic balance and seem harmonic and aesthetically pleasing. The golden ratio or the golden (divine) proportion is considered the most pleasing to human visual sensation and is not limited only to aesthetic beauty but also found prominently in the natural world. It is an irrational number denoted by ϕ and is approximately 1.618034. It is a fascinating topic for mathematicians, artists, natural scientists, philosophers, biologists, artists, architects, musicians, psychologists, designers, etc.

This research work presents a panoramic view of the golden ratio; their construction algorithms, and structures; mathematical presentations, geometry, patterns, and prop- erties from mathematical perspectives and opens the broad horizon for further research. Besides these, the paper also presents their existences, diversities, their relationships with nature, the universe, arts, design, architecture, and engineering in ancient and modern mathematics and sciences.

Keywords: Golden ratio, construction, geometry, beauty, diversity.

1 INTRODUCTION

The golden ratio is found abundantly in nature. It has been used for centuries in design, architecture, structure, and construction. If someone goes down deep enough into anything he/she will find mathematics. To find the mathematical beauty and joy in everything, one has to play with the divine proportion, the golden ratio. The golden ratio has been used not only in an ancient and classical structure but in modern architectures, artwork, and photography. It is found in nature, the universe, and in different aspects of mathematical sciences. Fibonacci numbers and the Fibonacci sequence have a close connection with the golden ratio and are also considered as nature's number system and are dominant. The golden ratio is one of the fascinating topics. Mathematicians since Euclid have studied it.

A Greek sculptor, Phidias (490-439 BC) used such a ratio in the design of most of his sculp- tures for different Parthenon. Later Plato, a Greek philosopher, implied it to the proportional relationship in his work Timaeus. However, golden ratio was defined in Euclid's Elements like, *a straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less,* as illustrated in Figure 1. For details, [1, 10].

Pythagoras theorem and the golden ratio had given great importance in the history of mathmematics, as Johannes Kepler also said, *Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into mean and extreme ratio. The first we may compare to a mass of gold, the second, we may call a precious jewel,[3].*

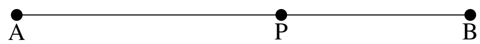


FIGURE 1: A line segment to get the golden ratio.

Consider a line segment AB. Consider a point P on the line AB such that

$$\frac{AB}{AP} = \frac{AP}{BP}$$

Let AB= x and AP=1, then we get, $x = \frac{1}{x-1}$, which yields, $x^2 - x - 1 = 0$.

$$\therefore x = \frac{1 \pm \sqrt{5}}{2}$$

For the length measure, $x = \frac{1 \pm \sqrt{5}}{2}$, an irrational number denoted by phi (ϕ). For decimal representation, ϕ = 1.6180339887498948482 . . . . This ratio is known as the golden ratio. It is also named the divine ratio or divine proportion. In the book, La Divine Proportione, The Divine Proportion, Luca Pacioli wrote an in-depth study of the golden ratio, and had mentioned that *Like god, the divine proportion is always similar to itself, [10].*

In 1977, Paul S. Bruckman, published an amusing poem in [4] called *Constantly Mean*. Referring to the Golden Ratio as the *Golden Mean*, the historical verse is,

The golden mean is quite absurd;
It's not your ordinary surd.
If you invert it (this is fun!),
You'll get itself reduced by one;
But if increased by unity,
This yields its square, take it from me.

The golden ratio is one of the fascinating topics from ancient history. There has been a lot of work about its historical background and existence. However, its systematic overview from the mathematical beauty perspective, construction algorithms, their geometry and diversities is somewhat lacking. This paper covers the state-of-art on the golden ratio, their mathematical structures, construction, properties, and applications in diversified fields with possible examples and illustrations, like a panoramic view. The rest of the paper is as follows. Section 2 is about the properties of the golden ratio and its alternative forms. About its construction algorithms in Section 3 and the geometry concerning its different structures is in Section 4. Section 5 shows the diversities in golden ratio from ancient sculptural structures, modern designs, human organs, music, and nature. Finally, Section 6 concludes the paper.

2 PROPERTIES OF GOLDEN RATIO

Some of the greatest mathematical minds of all ages, Pythagoras, Euclid, Pisa, Kepler, to present-day mathematical scientists have spent endless hours over this divine ratio. Not only for the mathematicians it is in the common interest of biologists, artists, architects, musicians, psychologists, designers, etc. It has inspired thinkers of all disciplines like no other number in the history of mathematics and has different properties, [8].

2.1 The golden ratio ϕ is an irrational number:

Assume the contrary as $\phi = \frac{1 \pm \sqrt{5}}{2}$ is rational. Then by the closure of rational numbers we get, $2(\frac{1 \pm \sqrt{5}}{2}) - 1$ is also rational, as a contradiction, as the square root of a non-square natural number is always an irrational number. Hence, the golden ratio ϕ is an irrational number.

2.2 The conjugate of golden ratio is the silver ratio:

The reciprocal of $\phi = \frac{1}{\phi} = \Phi$(say), as the golden ratio conjugate or the silver ratio. Hence, $\Phi = \frac{1}{\phi} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2} = 0.618033987498948 = 1.618033987498948 - 1 = \phi - 1$. Thus,the golden ratio has a unique property among the positive numbers as $\frac{1}{\phi} = \phi - 1$.

2.3 Alternative forms of golden ratio:

- **In the form of continued square root:**
  From $\phi = 1 + \frac{1}{\phi}$ we get,

$$\phi^2 - \phi - 1 = 0 \tag{1}$$

Its positive root is given by,

$$\phi = \sqrt{1+\phi} \tag{2}$$

Substituting successively the left hand side for ϕ on the right side of Equation 2 gives,

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+...}}}} \tag{3}$$

as in the form of continued square root.
Replacing $\phi = \frac{1}{x}$ in Equation 1 gives $x^2 + x - 1 = 0$ with its positive root as,

$$x = \sqrt{1-x} \tag{4}$$

Substituting successively the left hand side for x on the right side of Equation 4 gives,

$$x = \sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\dots}}}} \tag{5}$$

$$\therefore \frac{1}{\phi} = \sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\dots}}}} \tag{6}$$

as in the form of continued square root.

- **In the form of continued square root:**
  The golden ratio can be expressed as, $\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1+\frac{1}{\phi}} = ...$, repeatedly. It can be expressed in the form of continued fraction as follows:

$$\phi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+...}}} \tag{7}$$

Where its reciprocal can be expressed as

$$\phi^{-1} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+...}}} \tag{8}$$

Also, any power of ϕ is equal to the sum of two immediately proceeding powers as

$$\phi^n = \phi^{n-1} + \phi^{n-2} \tag{9}$$

If $n = 1$ then this Equation 9 reduces to $\phi = 1 + \frac{1}{\phi}$.

- **In the form of trigonometric ratios:**
  The golden ratio ϕ can be expressed in different trigonometric ratios like

$$\phi = 1 + 2sin(\pi/10) = 2cos(\pi/5) = 2sin(3\pi/10) = \frac{1}{2}cosec(\pi/10) \tag{10}$$

- **In the form of golden mean series:**
  It can also be expressed in the form of infinite series as the golden mean series

$$\phi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{4^{2n+3}n!(n+2)!} \tag{11}$$

3 CONSTRUCTION OF GOLDEN RATIO

Here, we are presenting two simple algorithms for the construction of golden ratio corresponding to the interior and exterior division on a line segment, respectively including their constructions.

Algorithm 1: A simple algorithm for the construction of ϕ with interior division.

1. Draw a line segment AB.
2. Draw DB perpendicular to AB.
3. Take a point C on BD such that $BC = AB$.
4. Draw CB=CP.
5. Draw AP=AQ.

Output: Golden ratio, $\phi = \frac{AQ}{PQ}$

Algorithm 2: A simple algorithm for the construction of ϕ with interior division.

1. Draw a line segment MX.
2. Draw ZX perpendicular to MX with MX=ZX.
3. Take N as the midpoint of MX.
4. Draw a circle with center N and radius NZ which meets MX produced at Y.

Output: Golden ratio, $\phi = \frac{MX}{XY}$

4 GEOMETRY OF GOLDEN RATIO

Here, we present the geometry of golden ratio corresponding to triangle, rhombus, rectangle, spiral, pentagon, pentagram, ellipse, pyramid, etc.

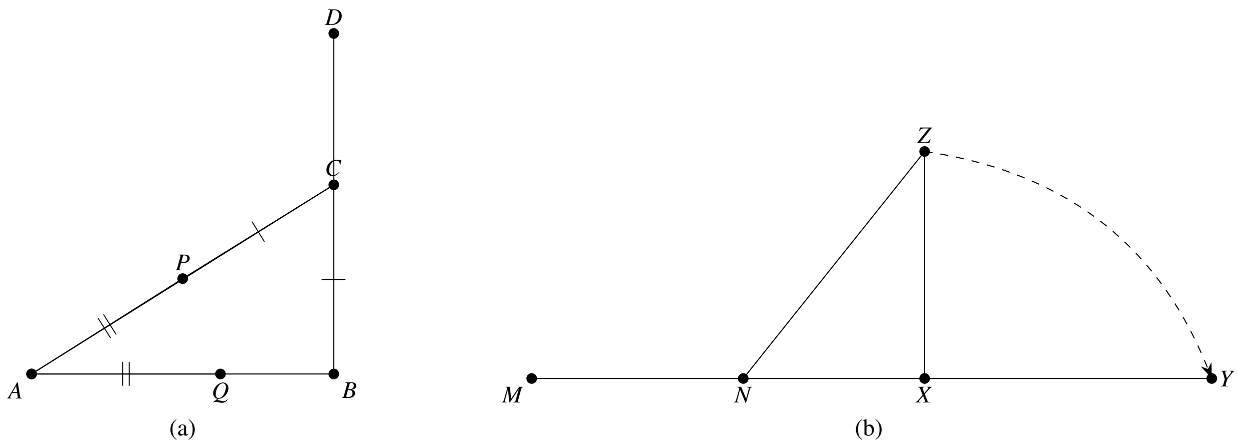


FIGURE 2: Construction of golden ratio ϕ with (a) interior division, $\phi = \frac{AQ}{BQ}$ and (b) exterior division, $\phi = \frac{MX}{XY}$

4.1 Golden triangle

Consider an isosceles triangle ABC with base angles $\frac{2\pi}{5}$. Here $\alpha = \frac{\pi}{5} = 36°$. Let the bisector of $\angle ABC$ meet AC at P.
Then,

$$\angle BPC = \pi - \frac{\pi}{5} - \frac{2\pi}{5}$$

Geometrically, $\triangle ABC \sim \triangle BCP$.
Let, CP has length 1, then length of BC is ϕ in the golden triangle ABC as in Figure 3.
Here, $AB : BC = BC : CP$.
Then,

$$\frac{AB}{\phi} = BC$$

It yields, $AB = \phi^2$ But, $AB = AC$.
Hence,

$$\phi^2 = \phi + 1$$

Such a cut by BP in $\triangle ABC$ is named as golden cut where $\triangle ABP$ and $\triangle BCP$ are golden gnomon and golden triangle, respectively.

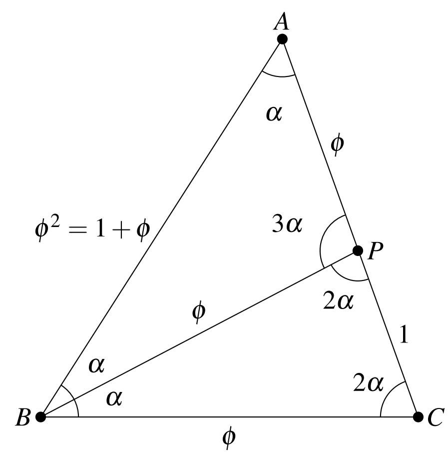


FIGURE 3: Golden triangle.

4.2 Alternative forms of golden ratio

A rhombus having its diagonals in golden ratios is the golden rhombus, as in Figure 4 (a). Here, in rhombus $EGFH$, the diagonals EG and FH are perpendicular to each other where $FH = 1$ and $EG = \phi$. Such a golden rhombus can be formed by joining the mid-points of the adjacent sides of a golden rectangle.

The golden rectangle is considered the most visually pleasing of all rectangles. It can be found in the shape of playing cards, windows, book covers, file cards, painting canvas, ancient buildings, and modern skyscrapers. The canvas of the world-famous painting Monalisa was also in a golden rectangle. A distinctive feature of such a rectangular shape is that when a square section is removed from it, the remainder remains a golden rectangle. Its construction algorithm is in Algorithm 3.

Algorithm 3: An algorithm to construct golden rectangle.

1. Construct a square ABCD on base AB.
2. Take a point M at AB such that AM=MB.
3. Draw a circle with centre M and radius MC.
4. Produce MB to intersect the circle at P.
5. Draw a perpendicular at P which meets DC produced at Q.

Output: Rectangle $APQD$ is the golden rectangle. ($BPQC$ is also a golden rectangle).

4.3 Golden spiral

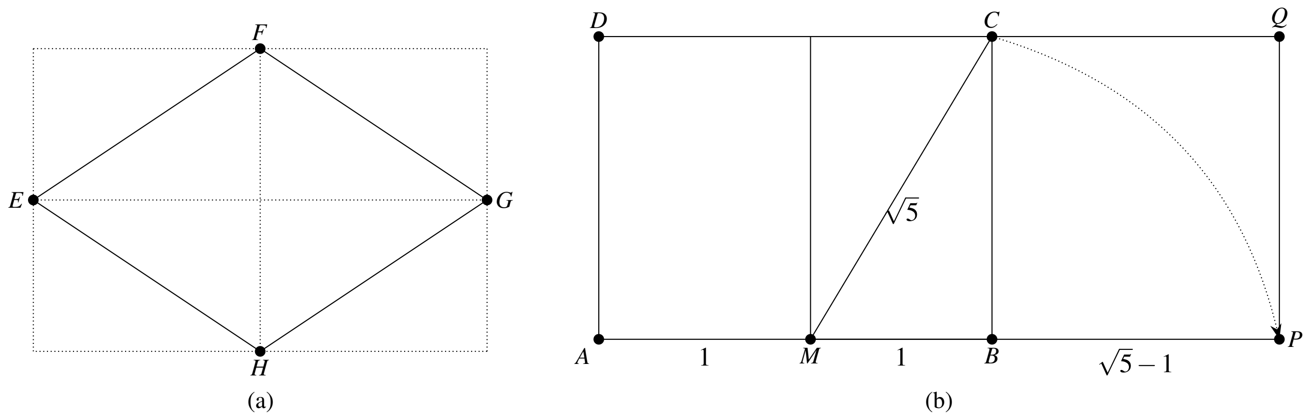


FIGURE 4: (a) Golden rhombus, (b) Construction of golden rectangle.

A golden spiral can be constructed iteratively on golden rectangles. Removal of squares has another golden rectangle, and can repeat it infinitely. Corresponding corners of the square form an infinite sequence of points on the unique logarithmic spiral called the golden spiral, as in Algorithm 4. It is illustrated in Figure 5.

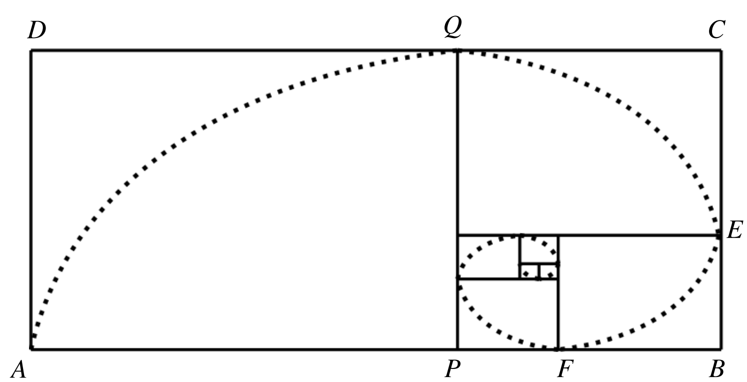


FIGURE 5: Golden ratio and golden spiral.

Algorithm 4: An algorithm to construct golden spiral.

1. Construct a golden rectangle ABCD on base AB.
2. Construct a square APQD by drawing a circle with radius equals height of rectangle.
3. Repeat Step 2 inside the smaller rectangle BPQC .
4. Produce MB to intersect the circle at P.
5. Inscribe quarter circles in each square to form spirals.

Output: A golden spiral shown with thick dotted lines in iterative golden rectangles.

Let AB = ϕ and AD=1. Then, QC = $\phi - 1 = \frac{1}{\phi}$. Hence, BE =$\frac{1}{\phi^2}$,PF = $\frac{1}{\phi^3}$, and so on. Thus for ϕ and 1 be the sides of a golden rectangle, sides of iterative squares inside such golden rectangle to form the golden spiral are with sides $1, \frac{1}{\phi}, \frac{1}{\phi^2}, \frac{1}{\phi^3}, \ldots$

4.4 Golden ratio from an equilateral triangle

Consider an equilateral triangle ABC. One can draw a circum-circle with the vertices of a triangle be on its circumference, as one and only one circle can be drawn passing through three non-collinear points, as in Figure 6. Consider P and Q to be mid points on AC and BC, respectively. Let PQ produced meets the circumference at R. Coxeter posed Odom's construction in the form, if an equilateral triangle is inscribed in a circle and the line segment joining the midpoints of two sides is produced to intersect the circle in either two points, then these three points are in golden proportion, [5]. More precisely, $PQ : QR = \phi$ and $PR : PQ = \phi$.

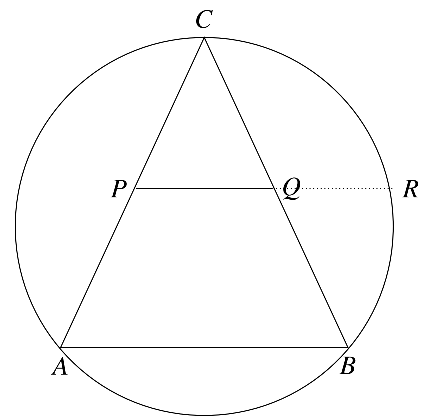


FIGURE 6: Golden ratio from an equilateral triangle.

4.5 Golden pentagon

The ratio of a side of a regular pentagon to its diagonal is a golden ratio. Hence, a regular pentagon is a golden pentagon. If a pentagon is divided diagonals from one vertex, the resulting triangles are golden. Among them, the middle one is the acute golden triangle. The other two are obtuse golden triangles. Let the side of the regular pentagon be 1. Then, in such a golden pentagon as in Figure 7, we get $DA = \phi$ and DF = $\frac{1}{\phi}$.

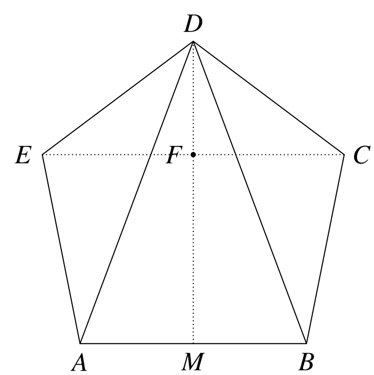


FIGURE 7: A regular pentagon as a golden pentagon.

4.6 Golden pentagram

Let a golden pentagram $ABCDE$. It has a regular inner pentagon $PQRST$ as in Figure 8. Such pentagram includes ten isosceles triangles. These can be catagorized as:

- Acute angled isosceles triangle:
  $\triangle ATP, \triangle BPQ, \triangle CQR, \triangle DSR$, and $\triangle EST$.
- Obtuse angled isosceles triangle:
  $\triangle PCE, \triangle QDA, \triangle REB, \triangle SAC$, and $\triangle TDB$.

In each, the ratio of longer side to the shorter side is ϕ. Acute angled isosceles traingles are the golden traingles and the obtuse angled isosceles traingles are golden gnomons.

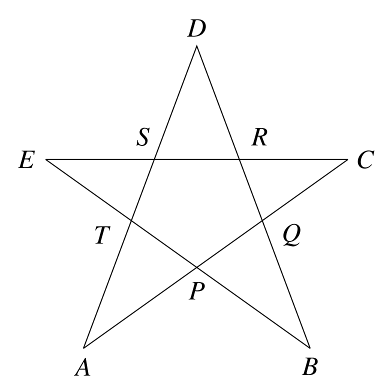
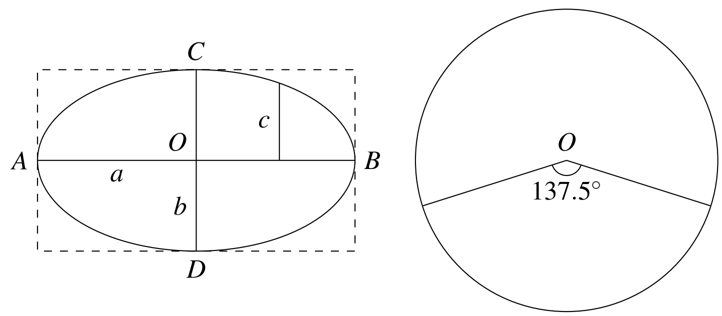


FIGURE 8: A golden pentagram.

4.7 Golden angle and golden ellipse

A circle can also be divided into two arcs in proportion to the golden ra-tio. For this, the smaller arc makes the corresponding central angle of 137.5 degrees and is considered as the golden angle,9 (b). A golden ellipse is an ellipse drawn inside the golden rectangle where it has theratio of its major axis to the mi- nor axis is in golden ratio 9 (a). A golden triangle is possible to draw inscribing in a golden ellipse that inscribes in a golden rectangle [7]. Let a, b, c, and e be the length of semi-major axis, length of semi-minor axis, the length of semi-latus rectum, and the eccentricity, respectively of a golden ellipse, then its eccentricity becomes, e = $\frac{1}{\sqrt{\phi}}$ where $a : b : c = \phi^2 : \phi : 1$.



(a) (b)

FIGURE 9: (a): A golden ellipse. (b): A circle with golden angle.

4.8 Golden pyramid

Consider a regular square pyramid with base $2b$, vertical height h and slant height (apothem) a as in Fig- ure 10. If $a : b = \phi$, it becomes the golden pyramid.
Here,

$$a = b\phi$$

Then,

$$h = \sqrt{\phi^2 - 1}.b = b\sqrt{\phi}$$

The ratio of the semi-base, vertical height and the apothem in a golden pyramid are given by:

$$b : h : a = b : b\sqrt{\phi} : b\phi = 1 : \sqrt{\phi} : \phi$$

Such a right angled triangle having the sides in the ratio of $1 : \sqrt{\phi} : \phi$ has been mentioned as a Kepler triangle [6] and it combines two key mathematical concepts as the Pythagorean theorem and the golden ratio.

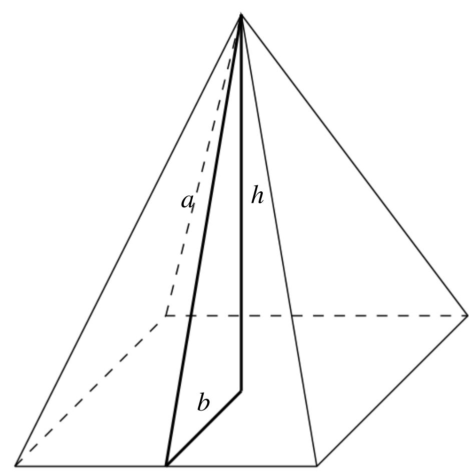


FIGURE 10: A regular square pyramid as the golden pyramid.

If the perimeter of the regular square base of the pyramid be 2π times its vertical height, then we get:

$$8b = 2\pi h$$

i.e.

$$h : b = \frac{4}{\pi}$$

Then in such a pyramid,

$$b : h : a = b : 4b/\pi : a$$

It is almost identical to golden pyramid with $a : b = \phi$ for what $b : h : a = 1 : 4/\pi : \phi$ with $\sqrt{\phi} \approx \frac{4}{\pi}$. Egyptian pyramids are very closed in proportion to such π-based mathematical pyramids.

Based on 3 : 4 : 5 triangle, Papyrus had described a nearly similar pyramid. Most of the Egyptian pyramid are with $b : h : a = 3 : 4 : 5$, as Eric Tempe Bell, claimed in 1950 that the only right triangle known to Egyptian at that ancient time was 3 : 4 : 5. They did not know the Pythagorean theorem, nor any way to reason about irrationals such as π or ϕ [2]. One of the Egyptian pyramid Giza, one of the seven wonders of the ancient world, is close to a golden pyramid. Moreover, it is even closer to π-based mathematical pyramid. Historians of science have long debated whether the Egyptian had any such knowledge, contending that its apearance in the great pyramid of Giza is the result of chance [9].

5 DIVERSITIES IN GOLDEN RATIO

Here, we are presenting the existence of the golden ratio in diversified fields like ancient architects/structures, modern designs, different organs of the human body, music, and nature.

The golden ratio had been used in most of the famous ancient architects and sculptural structures like the great pyramid of Giza, Parthenon. There remains a topic of controversy whether such an amazing relationship to the golden ratio in such Egyptian pyramids is by design or by accident as a result of chance. The Parthenon's facade can be circumscribed by golden rectangles. It indicates that the architects were aware of the golden ratio and employed it in their designs. It might be possible that they used their sense of good proportion in their design that closely approximates the golden ratio, [1]. Such a divine proportion is present not only in such ancient architects but in the world-famous modern structures like the Eiffel Tower of Paris and the Taj Mahal of Agra.

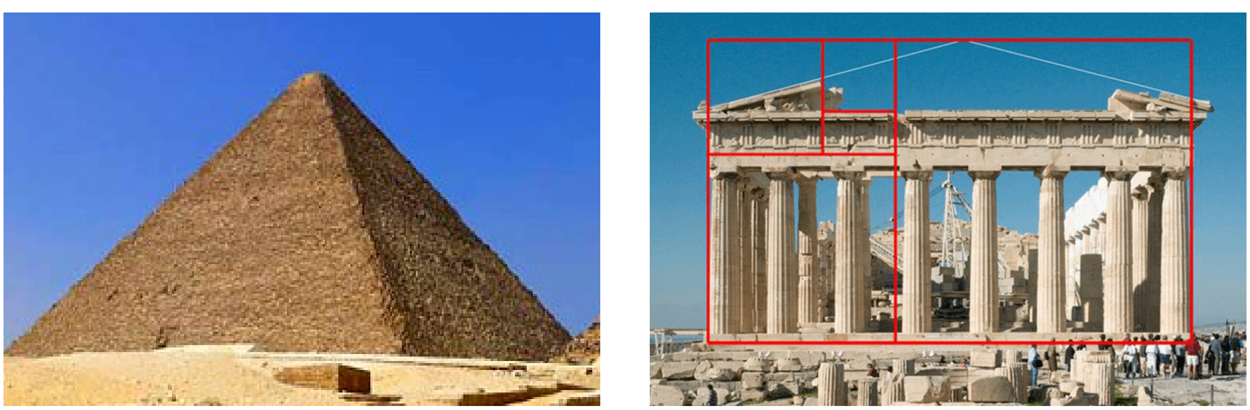


FIGURE 11: Golden ratios in some of the ancient and famous architects[1]

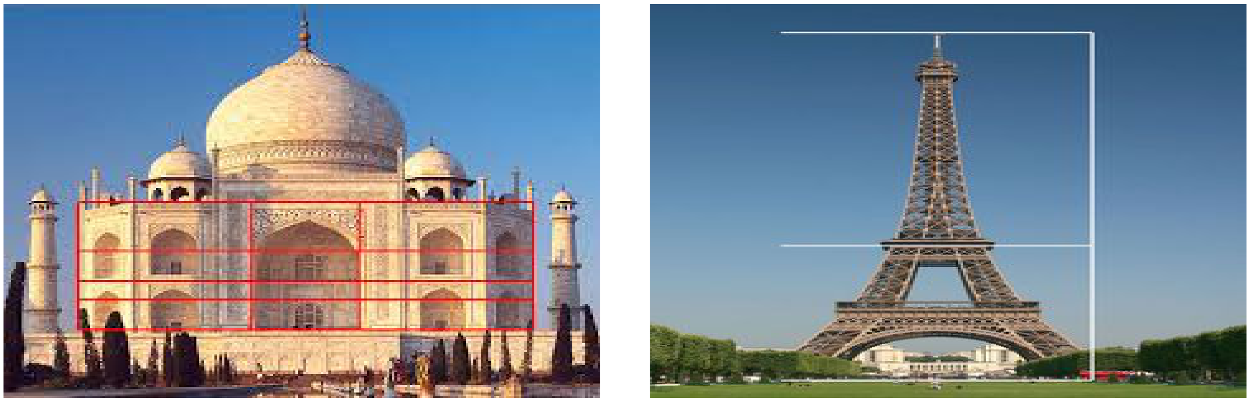


FIGURE 12: Golden ratios in some of the famous modern architects[2]

5.1 Golden ratio in modern design

The existence of golden sections and the divine ratios in inanimate and artistic endeavors is more pleasing and appealing to the human eye. It creates a feeling of satisfaction and harmony within an image. Golden sections and ratios are used extensively in modern designs and arts. For example, the Twitter logo and Apple iCloud logo open their mystery in their design strategies using golden rectangles, Fibonacci numbers, and the golden ratio. A simple golden rectangle is used as a logo to represent national geography. The Pepsi logo consists of two intersecting circles that are in a golden proportion to each other. Toyota's logo consists of three ovals. Two perpendiculars ovals combine to form like a 'T' which stands for Toyota. The ratio between the logo, both vertically and horizontally, is approximately ϕ, like the golden ratio. These two ovals, like consumers and the products, are maintaining like the divine proportion. Figure 13 shows how such sections and the golden ratios are applied to design such attractive logos.

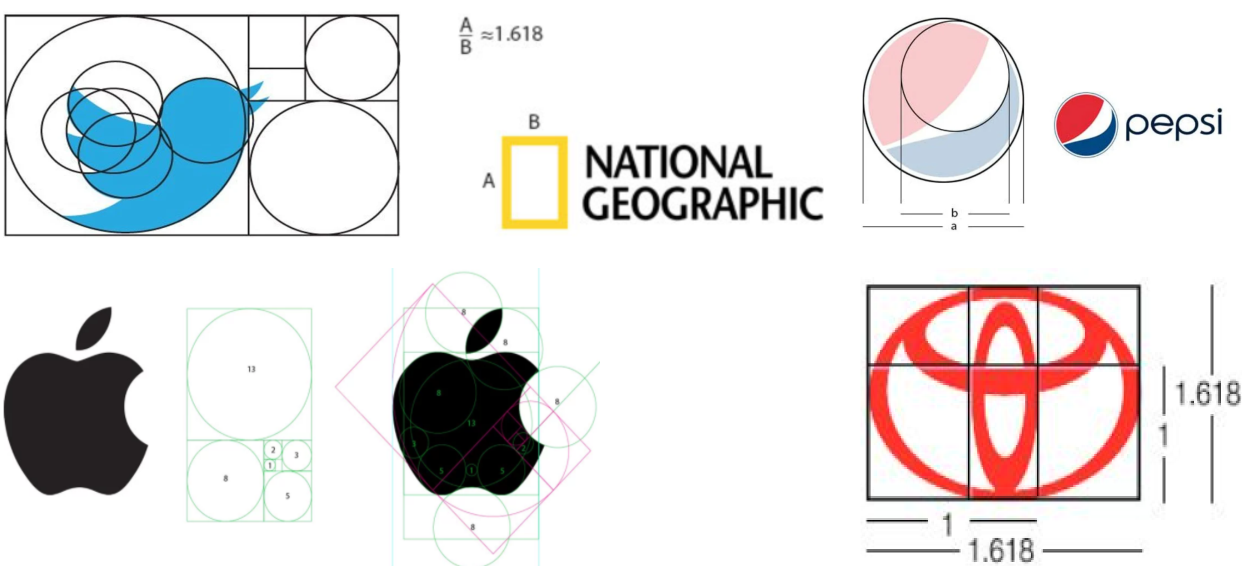


FIGURE 13: Logo designs of Twitter, National geographic, Pepsi, Apple, and Toyota.

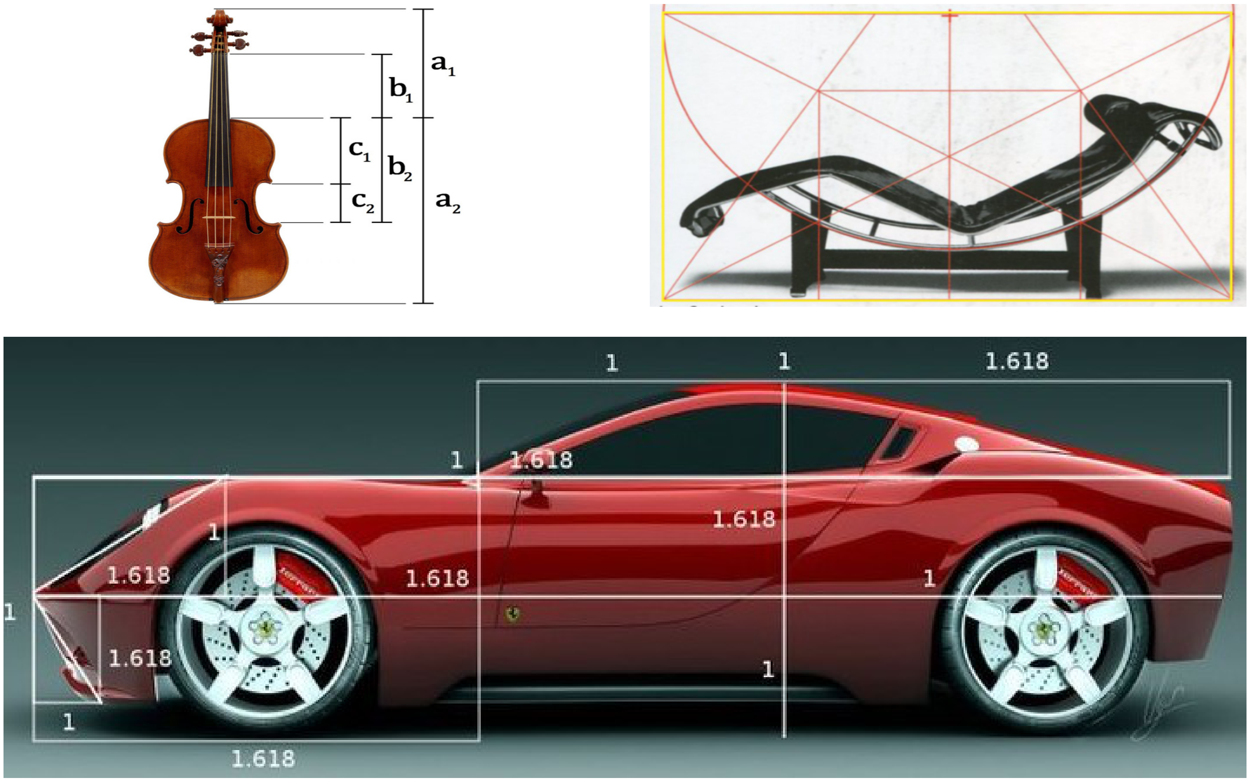


FIGURE 14: Attractive designs of a guitar, a rest chair, and a Toyota car, respectively.

The golden ratio is in the design of a violin. Each section of a violin follows the golden ratio in comparison to its other sections. Such proportions are for the best quality of sound projection for optimal performance. Chaise Longue, a long chair that is long enough to support the legs has the proportion of the chaise relating to the harmonic subdivision of a golden rectangle. The chair back fits perfectly into a golden section rectangle. In Toyota cars, such a ratio has been used in most of the body parts too for the pleasant looking as mentioned in [1].

5.2 Human body, golden ratio, and beauty

The relationship between the golden section and the divine proportion is present in various parts of a human body. For example, the total height of a body and the distance between head to fingertips; distance between head to naval and naval to hill; the bones of fingers in hand related to each other; fore-arm and upper arm; hand and forearm. It is a surprising fact that the human being is the most perfect instantiation of god within such divine proportions. Not only in such organs, but it is also found even in the human skull, [11].

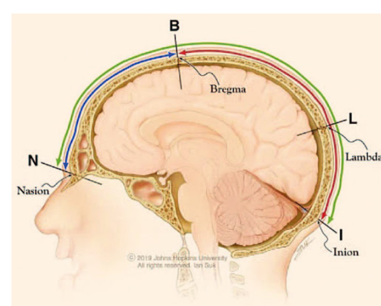


FIGURE 15: Human Skull.

Consider a human skull as in Figure 15. Let N, B, and I denote the Nasion, Bregma, and Inion, respectively. In human anatomy, Nasion(N) is the boundary between the nasal and frontal bones (frontonasal suture). Bregma (B) is the junction of the coronal and sagittal sutures, and Inion (I) is the external occipital protuberance. Consider the following three measures as

- Nasionic arc (NI): the surface distance between N and I
- Frontal arc (NB): the surface distance between N and B
- Parieto-occipital arc (BI): the surface distance between B and I

These two ratios $NI : BI$ and $BI : NB$ both are approximately the golden ratio ϕ, as mentioned in [11]. The authors studied in different mammalian species and observed that these two ratios were not only different from ϕ but also unique to each species.
Human faces are also with this proportion within the relationships between the ears, eyes, mouth, and nose, as noticed in Figure 16. Such a divine proportion as well as the golden rectangles are prominent in most of the parts of a human body as shown in Figure 17.

FIGURE 16: Ear, teeth, and face of human body, respectively(from left).

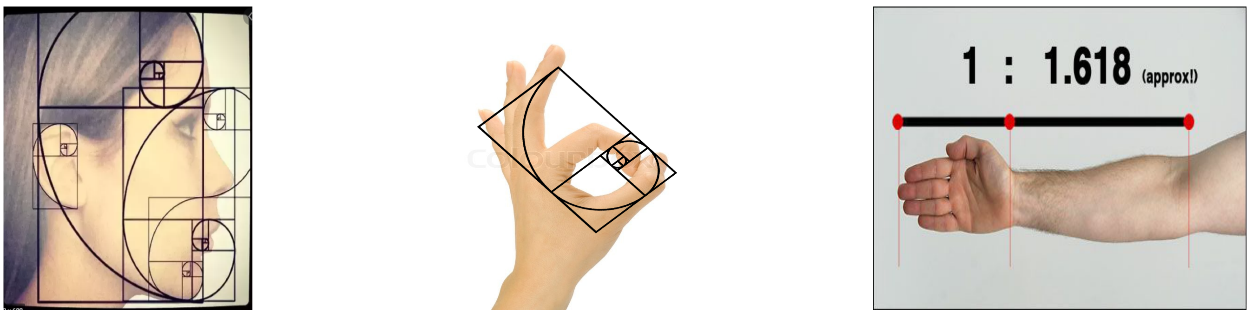


FIGURE 17: Human organs with golden ratios.

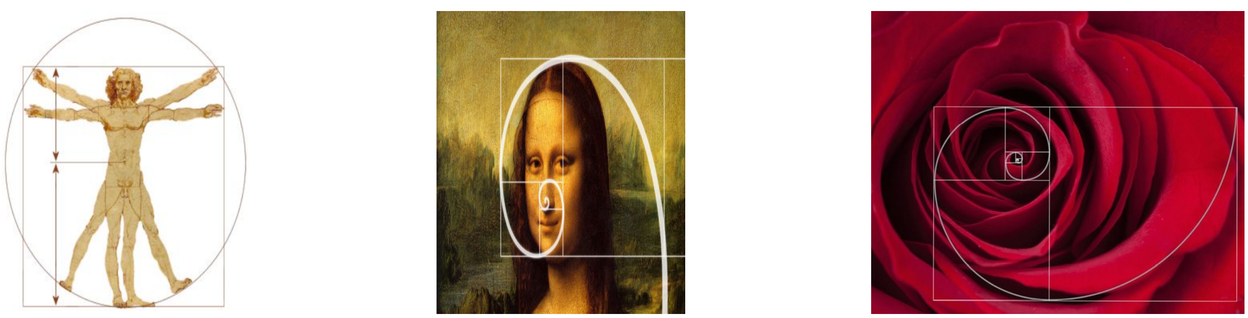


FIGURE 18: Vitruvian, Monalisa painting, and a rose flower(from left).

Our attraction to another person's body increases if that body is symmetrical and in divine proportion. Likewise, if a face is in divine proportion, we are more likely to notice it and find it beautiful, pleasant, and appealing. Science believes that the perceived proportional bodies are more healthy. This is also suggested centuries before by the famous artist Leonardo da Vinci showing an idealized human body within a square and a circle. However, in his Vitruvian Man image, the respective golden ratio like the ratio of the distance from foot to naval is to the distance from naval to head is approximately 1.505:1, and is not so close to 1.618:1, and was considered as a picture of perfection.

5.3 Golden ratio and music

Let a song lasts for 4 minutes. Then it is divided into two parts at 61.8% and 38.2% with a modification with a certain change, a bridge, or an arrangement with a different instrument or a new melodic composition at 61.8% of 240 seconds, i.e. at 148.32 seconds. The changes in the rhythm of the song are followed on the Fibonacci sequence, to make it more attractive and lovely. Great composers of western music like Beethoven, Mozart, and Wagner intentionally changed the rhythm of their music in sequences.

There are 13 notes in the span of any notes through its octave. The word octave refers to the eight whole tones of the complete musical scale. The dominant note is the 5th note of the major scale that is the 8th note of all the 13 notes on the octave. The key frequencies of the musical notes are related to 1, 1, 2, 3, 5, and 8, as the Fibonacci numbers. Musical compositions often reflect these numbers and the golden ratio on their timing. Musical notes progress in high and low pitches with an infinite spiral in the same manner as the golden spiral. The golden ratio is the mathematical translation of an algorithm used by nature and is a lesson on aesthetic perfection, beauty, harmony, and pleasure to the music too. The golden ratio is also used in the design of various musical instruments like piano, violin, guitars, and even in the high-quality speaker wire. For details, we refer to [1, 11].

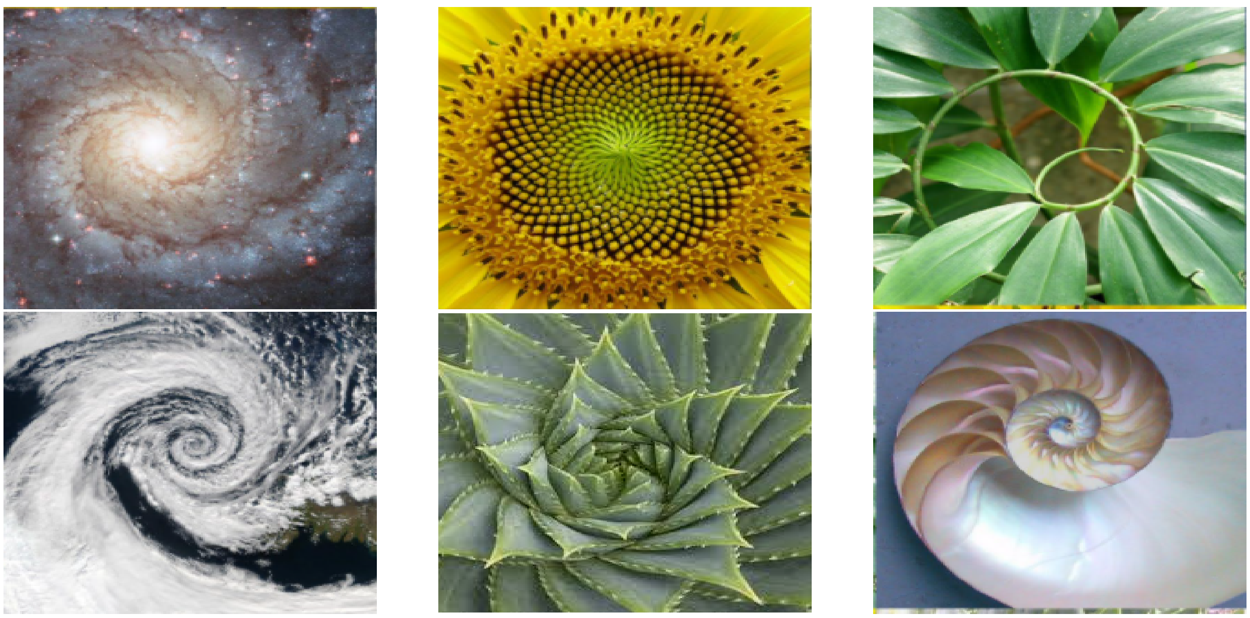


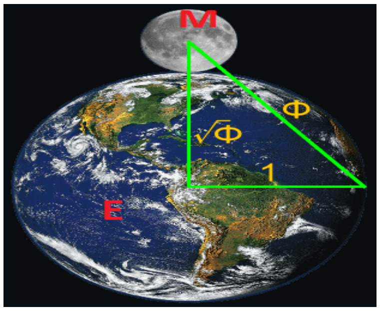
FIGURE 19: Nature is the great mathematician to design golden ratio.

Not only in ancient architects but nature itself follows the Golden ratio. It is present in various flowers, seeds, seashells, honeycombs, etc. For example, as in Figure 19, we have the Galaxy, florets of a sunflower, pine leaves, hurricane structure, spiral aloe leaves, and a nautilus shell in order. All these spirals can be obtained from the spirals made by golden rectangles. Nature itself is a great mathematician to design the golden ratio.

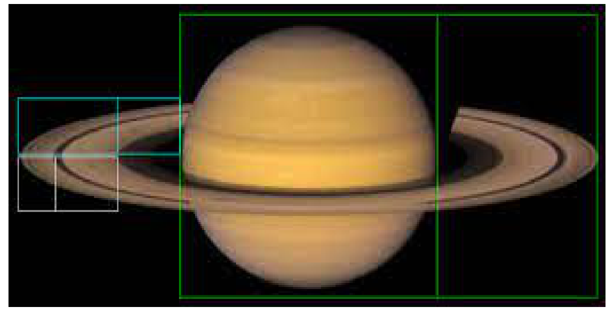
Let E and M denotes the earth and the moon, then their dimensions are also in ϕ, assuming the horizontal radius of the earth be 1 unit as in Figure 20 (i). The shape of the universe itself is a dodecahedron based in ϕ. Certain planets of our solar system are closely related to ϕ. For example, the planet Saturn reveals ϕ relationship in several of its dimensions as mentioned in Figure 20 (ii).

6 CONCLUSION

The golden ratio ϕ is one of the world-famous astonishing numbers. It is recognized for its ability to give a sense of aesthetic appeal in the beauty, balance, and harmony of the design. Such a ratio is used to add style and appeal to the marketing and advertisement of everyday consumer products. Many product logos of different famous companies and production houses, which represent a memorable image on the conscious and subconscious minds of consumers are also designed by using golden ratios. The golden ratio becomes a golden spiral in its interconnection with the Fibonacci sequence. Such a divine spiral plays an enigmatic role everywhere in nature.



(I) The Earth and the Moon



(II) The Saturn

FIGURE 20: The Earth, Moon and Saturn

This paper represents a qualitative overview of the golden proportion from ancient times to the modern age in connection to their nature as well as the physical world. It covers the state-of-art on the golden ratio, their mathematical structures, formulations, construction, properties, and applications in diversified fields with possible examples and illustrations, like a panoramic view, including their impacts on beauty and harmony. It provides the mystery of various geometrical patterns, their topological structures, and their construction algorithms from mathematical perspectives and opens the broad horizon for further research.

REFERENCES

1. Akhtaruzzaman, M., and Shafie, A.A. (2011) Geometrical Substantiation of Phi, the Golden Ratio and the Baroque of Nature, Architecture, Design, and Engineering. International Journal of Arts, 1(1), 1-22.
2. Bell, E. T. (1940) The Development of Mathematics. New York.
3. Boyer, C. B. (1968) A History of Mathematics, Wiley, New York.
4. Bruckman, P.S. (1977) Constantly Mean. https://pleacher.com/mp/mpoetry/goldmean.htm
5. Doris S. (2006) Coxeter and the Artists: Two-way Inspiration. In Harold Scott MacDonald Coxeter (ed.), Chandler Davis (ed.), Erich W. Ellers (ed.): The Coxeter Legacy: Reflections and Projections. American Mathematical Society, 2006.
6. Herz-Fischler, R. (2000) The Shape of the Great Pyramid. Wilfrid Laurier University Press, Waterloo.
7. Huntley, H.E. (1964) The Golden Cuboid, Fibonacci Quarterly 2 (3), 184-.
8. Livio, M. (2002) The golden ratio: the story of phi, the world's most astonishing number, Broadway Books, New York.
9. Markowsky, G. (1992) Misconceptions about the Golden Ratio. College Mathematics Journal, Mathematical Association of America, 23 (1), 2-19.
10. Pacioli, L. (1509) De divina proportione (English: On the Divine Proportion), (Antonio Capella) Venice: Paganino Paganini.
11. Tamargo, R.J., and Pindrik, J. A. (2013) Mammalian Skull Dimensions and the Golden Ratio ϕ, The Journal of Craniofacial Surgery, 30(6), 1750-1755.