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Research Management Cell, Pokhara Multiple Campus, Pokhara, Nepal
Email: research.intelligencepmcrmc021@gmail.com
URL: www.pmcpokhara.edu.np

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An Insight on the Geometry of the Golden Ratio

[1]Iswar Mani Adhikari

Abstract

Mathematics has a strong logical interconnection with structural design and architecture. Ratios are good examples of their interconnectivity. Nature is attractive due to the proper ratios of various components in them and between others. In nature, trees, forests, leaves, flowers, fruits, Himalayas, hills, valleys, springs, rivers, lakes, etc. show a rhythmic balance and seem harmonic and aesthetically pleasing. The golden ratio is considered the most pleasing to human visual sensation not limited only to aesthetic beauty but also found prominently in the natural world and is a fascinating topic. It can be constructed in different ways depending on the geometrical structures preferred. In this paper, our focus is on the geometry of the golden ratio. .

Keywords
Golden ratio
Construction
Geometrical structures

THE GOLDEN RATIO has been used for centuries in design, architecture, structure, and construction. It has been used not only in ancient and classical structures but also in modern architecture, artwork, and photography. It is found in nature, the universe, and various aspects of mathematical sciences. The golden ratio is one of the fascinating topics. Mathematicians since Euclid have studied it. Mathematics theorem and the golden ratio have been given great importance in the history of Mathematics, as Johannes Kepler also said, Geometry has two great treasures: one is the theorem of Pythagoras, and the other is the division of a line into mean and extreme ratios. The first we may compare to a mass of gold, the second, we may call a precious jewel. For details, we refer to (Bell, 1940; Boyer, 1968; Herz-Fischler, 2000; and Pacioli, 1509).

There has been a lot of work about its historical background and existence. However, its systematic overview from the geometrical

1. Associate Professor, Department of Mathematics, Tribhuvan University, Prithvi Narayan Campus, Pokhara.
Email:ishwor.adhikari@prnc.tu.edu.np/adhikariim35@gmail.com

beauty perspective, and construction design is somewhat lacking. This paper covers the state-of-the-art golden ratio based on its mathematical structures and their constructional properties instead of its mathematical properties. The rest of the paper is as follows. Section 2 is about the geometry of the golden ratio in plane geometry and Section 3 is in solid geometry. Finally, Section 4 concludes the paper.

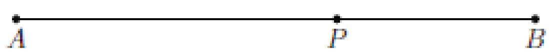
The golden ratio in Plane Geometry

Here, we are presenting the golden ratio corresponding to plane geometry. For details, we refer to (Akhtaruzzaman & Shafie, 2011; Livio, 2002; and Markowsky, G. (1992).

1 The golden ratio corresponds to a line segment

A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less, as illustrated in Figure 1.

Figure 1
The golden ratio in a line segment



Algorithm 1. Construction in a line segment

1. Consider line segment AB, Figure 1.
2. Consider a point P on the line AB such that AP=1 unit and $\frac{AB}{AP} = \frac{AP}{BP}$.
3. Let AB=c and form a quadratic equation from Step 2, i.e., $c^2 - c \quad 1 = 0$.
4. Solve the equation to get a real and positive root only, i.e., $c \quad \frac{1+\sqrt{5}}{2}$.
5. Such an irrational number in Step 4 is a golden ratio and is denoted by ϕ

Being an irrational number, it has non-repeating, non-terminating, and non-recurring decimal representation, like ϕ =1.6180339887498948482... This ratio is also known as the divine ratio or divine proportion. Here, we are using the term golden ratio

1.1 The golden ratio corresponds to internal division

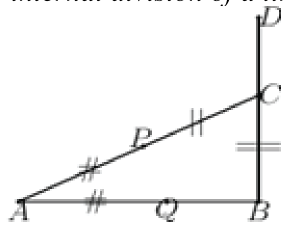
The golden ratio can be constructed corresponding to the internal division of a line segment.

Algorithm 2 Construction corresponds to the internal division in a line segment

1. Draw a line segment AB, Figure 2.
2. Draw DB perpendicular to AB.
3. Take a point C on BD such that BC=$\frac{1}{2}$ AB.
4. Draw CB=CP and draw AP=AQ.
5. Golden ratio, $\phi \quad \frac{AQ}{BQ}$.

Figure 2

The golden ratio from the internal division of a line segment



2 The golden ratio corresponds to exterior division

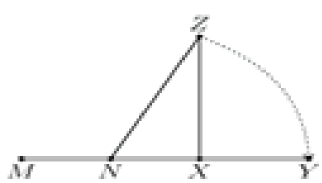
It can also be constructed in the form of the external division of a line segment.

Algorithm 3 Construction with exterior division of a line segment.

1. Draw a line segment MX, Figure 3.
2. Draw ZX perpendicular to MX with MX=ZX.
3. Take N as the midpoint of MX.
4. Draw a circle with center N and radius NZ which meets MX produced at Y.
5. Golden ratio, ϕ $\frac{MX}{XY}$.

Figure 3

The Golden Ratio corresponds to the external division of a line segment



2 The golden ratio corresponds to different triangles

It can also be defined as corresponding to an isosceles triangle and an equilateral triangle:

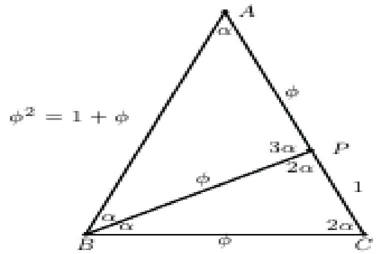
2.1 The golden ratio corresponds to isosceles triangles

Algorithm 4. Construction corresponding to an isosceles triangle

1. Draw an isosceles triangle ABC with $\angle ABC$ $\angle BCA$ $\frac{2\pi}{5}$, Figure 4.
2. Bisect $\angle ABC$ that meets AC at P.
3. Use geometry to calculate $\angle BPC$ $\frac{2\pi}{5}$.
4. $\Delta ABC \Delta BCP$ as their angles are $\frac{\pi}{5}$, $\frac{2\pi}{5}$, and $\frac{2\pi}{5}$, respectively.
5. Let CP=1, then BC= ϕ where, $\frac{AB}{BC} = \frac{BC}{CP} \Rightarrow AB$ ϕ .
6. It gives golden triangle ΔABC having sides 1, ϕ and ϕ .

Figure 4

Construction of a golden cut, golden gnomon, and golden triangle



Note that, such a cut BP in ΔABC is the golden cut where triangles ΔABP and ΔBCP are the golden gnomon and the golden triangle, respectively, Akhtaruzzaman & Shafie, 2011.

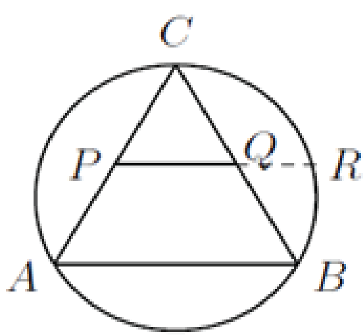
2.2 The golden ratio corresponds to an equilateral triangle

Algorithm 5 Construction corresponding to an equilateral triangle

1. Draw an equilateral triangle ABC, Figure 5.
2. Draw a circum-circle with the vertices of a triangle on its circumference.
3. Consider P and Q to be midpoints on AC and BC, respectively.
4. Let the PQ produced meet the circumference at R.
5. Then, the golden ratio, $\phi \quad \frac{PQ}{QR} = \frac{PR}{PQ}$.

Figure 5

Construction corresponding to an equilateral triangle.



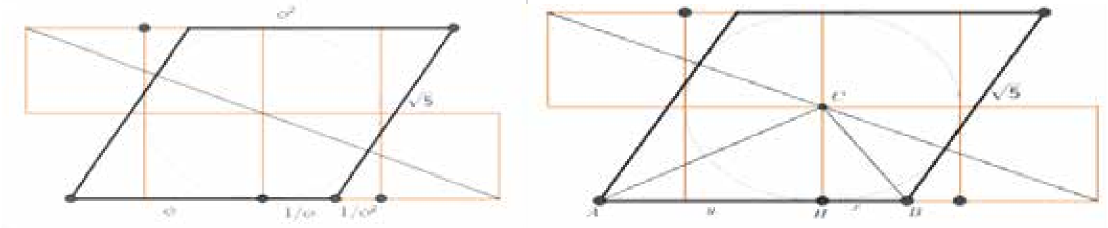
3. The golden ratio corresponds to different quadrilaterals

Here, we are presenting its geometry corresponding to different variants of the quadrilaterals, Akhtaruzzaman & Shafie, 2011.

3.1The golden ratio corresponds to a rhombus

Figure 6

Construction of golden ratio relative to rhombus



Algorithm 6 Construction corresponding to a rhombus

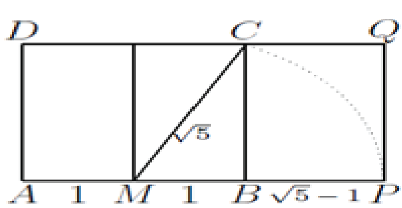
1. Put two 1×3 rectangles shifted one unit relative to each other, left part of Figure 6.
2. Draw the long diagonal of the resulting shape and the unit circle at its center.
3. Draw the tangents to the circle at the points where it meets the diagonal.
4. All sides of the obtained quadrilateral equal is $\sqrt{5}$, a rhombus.
5. Draw its (perpendicular) half-diagonals to obtain a right triangle ABC with the altitude CH, as in the right part of Figure 6.
6. Let AH=y and BH=x. Then xy=1, where $x \quad \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}$ and $y \quad \frac{\sqrt{5}+1}{2} = \phi$
7. Finally, $\frac{AC}{\,} = \sqrt{\frac{y+1}{x+1}} = \sqrt{\frac{\phi+1}{\frac{1}{\phi}+1}} = \phi$
8. The ratio of the diagonals in such a rhombus is ϕ and is a golden rhombus.

3.2 The golden ratio corresponds to a rectangle

The golden rectangle is considered the most visually pleasing of all rectangles. It can be found in the shape of playing cards, windows, book covers, file cards, canvases, etc. The canvas of the world-famous painting Monalisa was also in a golden rectangle. A distinctive feature of such a shape is that when a square section is removed from it, the remainder remains a golden rectangle, Meisner, 2018.

Figure 7

Construction of golden ratio relative to rectangle.



Algorithm 7 Construction corresponding to a rectangle

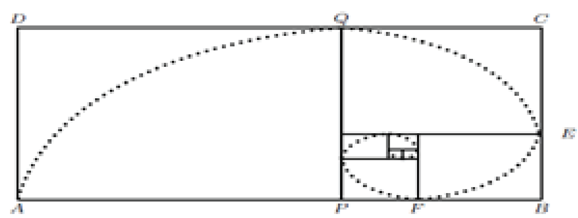
1. Construct a square ABCD on base AB, Figure 7.
2. Take a point M at AB such that AM=MB.
3. Draw a circle with center M and radius MC.
4. Produce MB to intersect the circle at P.
5. Draw a perpendicular at P which meets DC produced at Q.
6. Rectangle APQD is the golden rectangle. (BPQC is also a golden rectangle).

3.3 The golden ratio corresponds to a spiral (iterative rectangles)

A golden spiral can be constructed iteratively on golden rectangles. Removal of squares has another golden rectangle and can repeat it infinitely. Corresponding corners of the square form an infinite sequence of points on the unique logarithmic spiral called the golden spiral,

Figure 8.

Construction of golden spiral



Algorithm 8. Construction corresponding to a rectangle

1. Construct a golden rectangle on base AB, i.e., AB=ϕ and AD=1, Figure 8.
2. Construct a square APQD by drawing a circle with a radius equals height of a rectangle.
3. Repeat Step 2 in BPQC, then QC= ϕ -1= 1/ϕ . Hence, BE=1/ϕ^2 , PF=1/ϕ^3 , and so on.
4. Inscribe quarter circles in each square to form spirals.
5. A golden spiral is shown with thick dotted lines in iterative golden rectangles.

4. The golden ratio corresponds to different pentagons

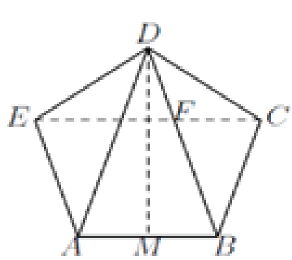
Here, we are presenting its geometry corresponding to the pentagon, Akhtaruzzaman & Shafie, 2011.

2.4.1 The golden ratio corresponds to a regular pentagon

Geometrically, the ratio of the diagonal of a regular pentagon to its side is a golden ratio. Hence, a regular pentagon is a golden pentagon. If a pentagon is divided into diagonals from one vertex, the resulting triangles are golden. Among them, the middle one is the acute golden triangle. The other two are obtuse golden triangles.

Figure 9

A regular pentagon is a golden pentagon



Algorithm 9 Construction corresponding to a pentagon

1. Construct a regular pentagon ABCDE on unit base AB, Figure 9.
2. Join its vertices to form diagonals, AD, BD, and BC.
3. Geometrically, DA=ϕ and DE=$\frac{1}{\phi}$

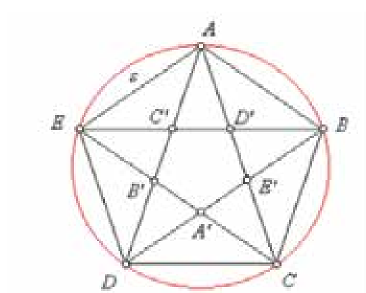
4.2 The golden ratio corresponds to a regular pentagram.

Diagonals of a regular pentagon form a regular pentagram. A regular pentagon is itself a golden pentagon. Such a pentagram is a golden pentagram. For details we refer to Meisner, 2018.

Algorithm 10. Construction corresponds to a pentagram.

1. Construct a regular pentagon ABCDE on base CD, Figure10.
2. Draw a circum-circle to this pentagon.
3. Draw all possible diagonals.
4. Geometrically, AC'EB'DA'CE'BD'A is a regular pentagram, and ithas a regular inner pentagon A'E'D'C'B' and ten isosceles triangles as:
   (i) Acute angled: ΔAC'D', ΔBD'E', ΔCE'A', ΔDA'B', and ΔEB'C'.
   (ii) Obtuse angled: ΔAC'E, ΔEB'D, ΔDA'C, ΔCE'B, and ΔBD'A.
5. In each, the ratio of the longer side to the shorter side is ϕ
6. Acute angled isosceles triangles are the golden triangles, and the obtuse angled isosceles triangles are golden gnomons.

Figure 10
A golden pentagram with golden polygon, triangles, and gnomons.



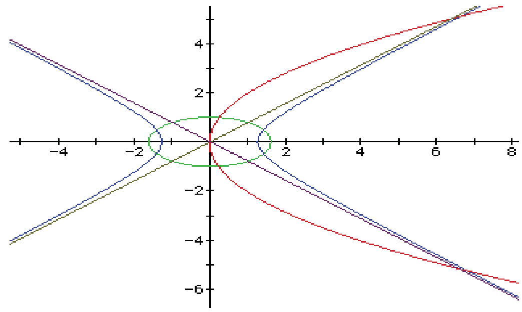
5. The golden ratio corresponds to different conics

Conic sections are the locus of a point thatt moves so its distance from a fixed point(focus) is a constant ratio (eccentricity, e) to the distance from a fixed line (directrix). The shape of such a curve is determined by this ratio e, where for the parabola, e=1; for the ellipse, e<1; and for the hyperbola, e>1. Consider the eccentricities of the ellipse and hyperbola are the golden sections, a:b= ϕ Parabola: $y\ = 4x$ Ellipse: $\frac{x}{\phi}+\frac{y}{1}=1$; and Hyperbola: $\frac{x}{\phi}-\frac{y}{1}=1$.

1. The latus rectum of the parabola is the directrix of the hyperbola.
2. The directrix of the parabola is the image in the **y**-axis of the directrix of the hyperbola.
3. The hyperbola asymptotes intersect the parabola at the points (4ϕ 4$\sqrt{\phi}$; (-4ϕ $-4\sqrt{\phi}$

Figure 11

Graphs of golden conics



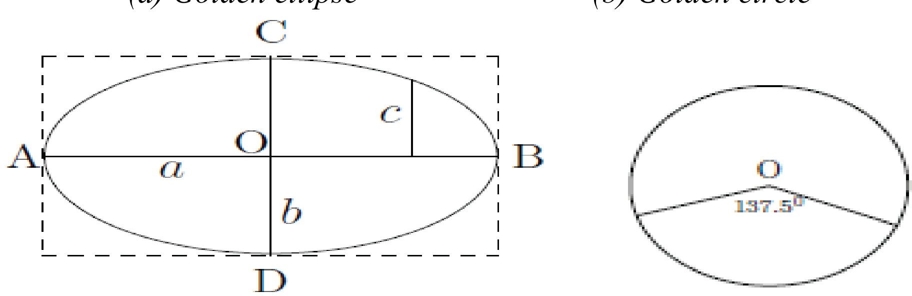
Algorithm 11 Construction of a golden ellipse.

1. Construct a golden rectangle with AB:CD= ϕ 2 (a).
2. Draw an ellipse with a and b as semi-major and semi-minor axis, as in
3. Such an ellipse is a golden ellipse $\frac{x}{a} + \frac{y}{b} = 1$, provided with a b ϕ
4. Let a, b, c, and e be the length of the semi-major axis, length of the semi-minor axis, the length of the semi-latus rectum, and the eccentricity, respectively of a golden ellipse, then its eccentricity, e $\frac{1}{\sqrt{\phi}}$ and a b c ϕ : ϕ 1.

Algorithm 12 Construction of a gold en circle

Figure 12

(a) Golden ellipse *(b) Golden circle*



Algorithm 13 Construction of a golden circle

1. Construct a circle in center O with an appropriate radius of users' choice.
2. Draw a central angle of $\frac{55\pi}{72}$ i.e., 137.5° at O as in Figure 12(b).
3. Geometrically, these two conjugate arcs are in proportion to the golden ratio and the angle $\frac{55\pi}{72}$ radian is called the golden angle in such a golden circle.

The Golden Ratio Corresponds to Solid Geometry.

The Golden ratio is a mystery and a matter of interest from ancient human civilization in different forms. It has a great connection in most of the beautiful solids . Here, we are presenting a few of such solids. For details, we refer to (Markowsky, 1992; Pacioli, 1509) .

3.1 The golden ratio in a golden pyramid

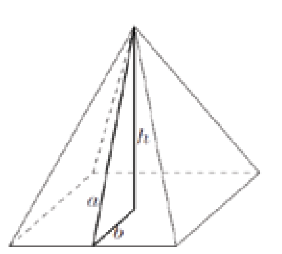
Algorithm 14. Construction algorithm of a g olden pyramid

1. Consider a square with base 2b, Figure 13.
2. Form a solid pyramid with vertical height a and slant height h such that $a \ b \quad \phi$
3. It becomes a golden pyramid where $b \ h{:}a \quad b \ b\sqrt{\phi} \ b\ \phi \quad 1{:}\ \sqrt{\phi} \ \phi$

Note that a golden pyramid combines two key mathematical concepts as the Pythagorean theorem and the golden ratio. If the perimeter of the regular square base of the pyramid be 2π times its vertical height, i.e. $8b \quad 2\pi$ then $b \ h{:}a \quad b \ \frac{4b}{}{:}a$ almost identical to golden pyramid with $a \ b \quad \phi$ Egyptian pyramids are very clos e to such pyramids. The Egyptian pyramid Giza, one of the seven wonders of the ancient world, is close to a golden pyramid.

Figure 13

A golden pyramid



2 The golden ratio in different structures

The golden ratio had been used in most of the famous ancient architects and sculptural structures like the great pyramid of Giza, Parthenon and even in the world-famous modern structures like the Eiffel Tower and the Taj Mahal, Figure 14 as in Livio, 2002; Meisner, 2018.

Figure 14

Pyramid of Giza, Parthenon, the Eiffel Tower, and the Taj Mahal.

3.2 The golden ratio in different modern design

The golden sections create a feeling of satisfaction, pleasure, and harmony within an image and are used extensively in modern designs and arts. For example, the Twitter logo, Apple iCloud logo, the Pepsi logo, Toyota's logo, etc. Moreover, an attractive design of a guitar, a rest chair, and a Toyota car, respectively, Figure 15 and 16. For details, Meisner, 2018.

Figure 15

The Twitter, Toyota lo0go, Pepsi Llogo and the Apple iCloud logo

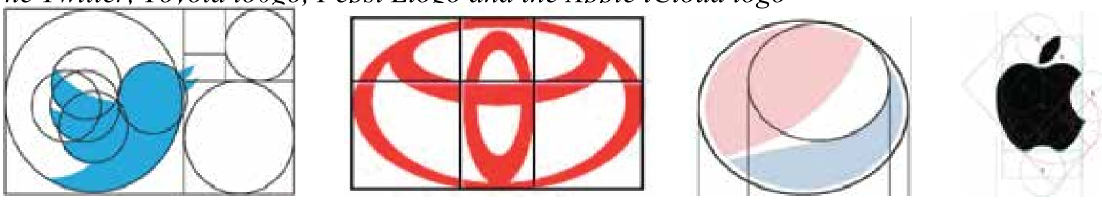


Figure 16

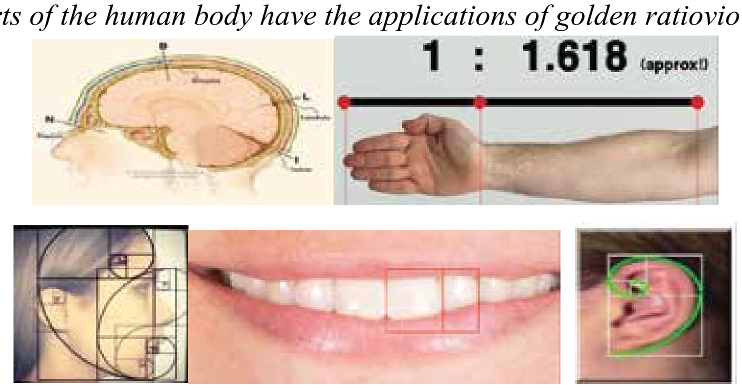
A violin, a Chaise Long and a Toyota Car

3.4 Natural construction of the golden ratio by God.

Various parts of a human body like the total height of a body and the distance between head to fingertips; distance between head to naval and naval to hill; the bones of fingers in hand related to each other; forearm and upper arm; hand and forearm, in most of the parts of skull, heart, lungs, liver, human face including ears, eyes, mouth, and nose have the applications of golden ratio in different forms, Figure 17. For details we refer to (Heneir, et al. , 2011; Tamargo, & Pindrik, 2013).

Figure 17

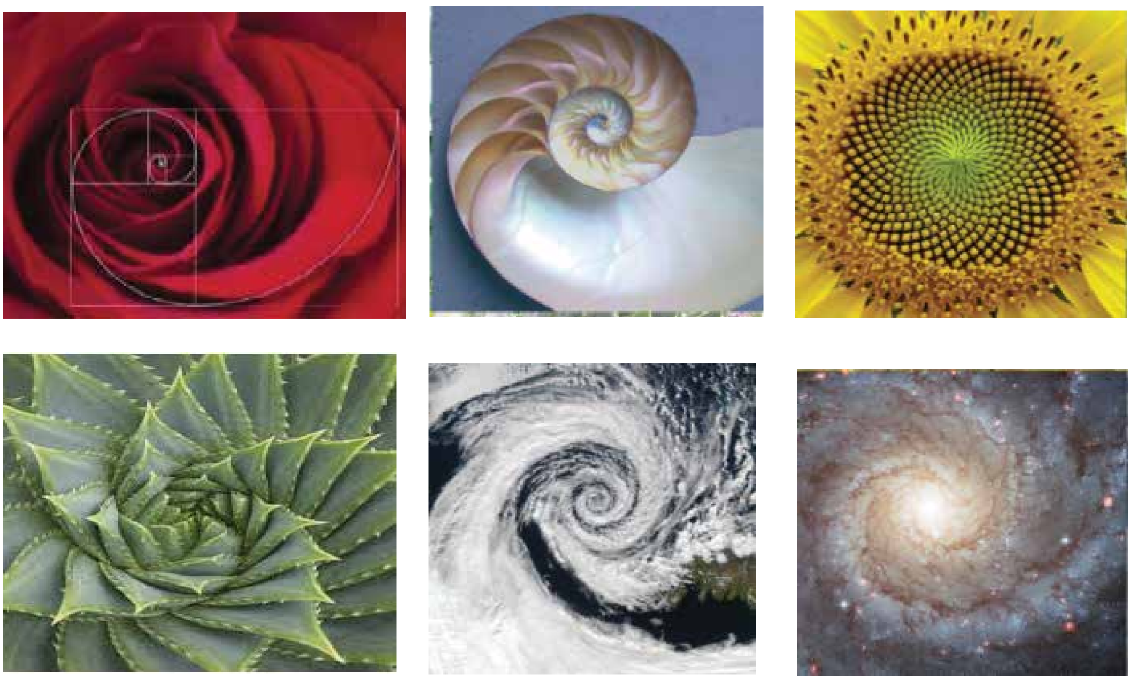
Different parts of the human body have the applications of golden ratiovio



Nature itself follows the Golden ratio in various flowers, seeds, fine leaves, seashells, honeycombs, etc. Certain planets of our solar system are closely related to the golden ratio. Not only these, the Galaxy, hurricane structure, spiral aloe leaves, and a nautilus shell in order from the spirals made by golden rectangles. Nature itself is a great mathematician to design the golden ratio.

Figure 14

Nature the great mathematicians to play with golden ratios



Conclusions

The golden ratio is one of the world-famous wondering and surprising numbers. It adds style and appeal to the marketing and advertisement of everyday consumer products to add beauty, balance, and harmony to the design. Such a property plays an enigmatic role everywhere in nature. This paper represents a qualitative overview of the golden ratio from ancient times to the modern age and covers the state-of-art on it fundamentally about their mathematical structures and construction algorithms in a lucid manner for the first time. It can be constructed corresponding to different geometrical shapes. It provides the mystery of various geometrical patterns, their topological structures, and their construction algorithms and their interconnections to nature.

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