JOL Established by INASP in 2007. Managed by Tribhuvan University Central Library.

Contents lists available at NepJol

Ganeshman Darpan

A Peer-Reviewed, Open Access Journal

Extension of Almost Contact Structure (ϕ_4,ξ_4,η_4) on $(N^{4n+3}\otimes R^d)\cong M^{5n+4}$

Nand Kishor Kumar [a, *]

[a]Lecturer, Trichandra Campus, Tribhuvan University, Nepal

Article Info **Abstract**

Received: January 2, 2024
Accepted: April 9, 2024
Published: July 22, 2024
Available online: August 10, 2024

DOI:https://doi.org/10.3126/gd.v9i1.68550

Keywords: *Almost contact structure, Diffeomorphism, Extension, Fibre bundle, Manifold, Algebraic structure.*

A prior research of almost contacts on 1, 2, and 3-manifolds has been partially investigated. On the other hand, the existence and geometry of a virtually contact 4-structure are poorly understood. A class of almost contact structures, g c d (2, n) = 1, has been created in this article. It is related to an almost compact 3-structure carried on a smooth Riemannian metric manifold (M,) of dimension (5n + 4). Using the almost contact metric manifold (,) as a starting point, we have demonstrated the existence of an almost contact structure on () from (for i = 1,2,3 on , constructed as a linear combinations.

1.Introduction

Contact geometry, discovered in 1896 by Sophus Lie, is a crucial tool for studying odd-dimensional manifolds and is closely related to various mathematical structures, with Gibbs' work on thermodynamics. It also has connections with fluid mechanics, Riemannian geometry, and low-dimensional topology (Geiges,1993).

Geiges discovered both (2n+1)-dimensional manifolds and 5-dimensional manifolds, but little was done on high-dimensional manifolds (Boothy &Wang,1958). Differentiable manifolds of contact and almost contact structures were typically classified (Borman, et.al. 2014). Adara studied the almost k-contact structure and identified an iso-parametric function similar to the construction by Mihai & Rosca,2005). He explained a manifold of (n+k+nk)-dimensional.

$(\phi_i,\xi_i,\eta_i),\cdots\cdots (\phi_k,\xi_k,\eta_k)$ and

$$\phi_i\circ\phi_j = -\delta_{ij} I_{\Gamma T} + \eta_i\otimes \xi_j + \sum_{I=1}^{k} \epsilon_{ij} I\phi I, \text{ and } \eta_i(\xi_j) = \delta_{ij} \qquad (1)$$

This concept was first used by Tachibana and Yu to demonstrate that the first three structures (ϕ_1,ξ_1,η_1), (ϕ_2,ξ_2,η_2), (ϕ_3,ξ_3,η_3) and the associated complex structures do not allow for the existence of a fourth almost contact structure (ϕ_4,ξ_4,η_4) with $\eta_i((\xi)_4)=\eta_4((\xi)_i) = 0$ for all i = 1, 2, and 3 (Adara,2009).

The objective of this article is to establish that given an essentially contact 3-structure, there is a structure (ϕ_4,ξ_4,η_4) that is dependent. This fourth structure's constructability contradicts Tachibana and Yu's hypothesis. The geometry of the manifold (ϕ_4,ξ_4,η_4) investigated (Tachibana & Yu,1970).

2.Statement of Problem

Nearly contact Riemannian manifolds significantly advance Geometry and Algebraic Geometry, leading to advanced applications in Mathematics, Computer Science, and Engineering due to unique tensor properties.

Geometers (Borman, et. al, 2014), (Kuo,1970), (Mihai & Rosca,2005), (Mehmet,2005) have explored almost contact 1,2,3-manifolds, but there is limited knowledge about it on an odd-dimensional manifold M.

Despite the fact that Kuo (1970) has shown a 3- almost contact structure on N^{4n+3}. With the anti-commutativity

* Corresponding author.
E-mail addresses: nandkishorkumar2025@gmail.com

Cite: Kumar, N. K. (2024). Extension of Almost Contact Structure (ϕ_4,ξ_4,η_4) on $(N^{4n+3}\otimes R^d)\cong M^{5n+4}$ *Ganeshman Darpan, 9*(1), 27-32. https://doi.org/10.3126/gd.v9i1.68550

condition on any odd dimensional manifold. Tachibana and Yu [6] conjectured that a fourth nearly contact structure cannot exist. In addition to creating a fourth almost contact structure, this research establishes the necessity for a third virtually contact structure (ϕ_4,ξ_4,η_4) on manifold $(M^{5n+3} \otimes R^d) \cong (N^{4n+4} \otimes R^d)$: $d|(2n+1)$ and g c d $(2, n) = 1$.

3.Objective

To make a 4- almost contact structure (ϕ_4,ξ_4,η_4) on $(N^{4n+3} \otimes R^d) \cong M^{5n+4}$ from

(ϕ_i,ξ_i,η_i) for i = 1,2,3 on N^{4n+3},

where g c d: $(4, d) = 1$, $(2, n) = 1$ and $d \mid (2n+1)$

4. Significance

The study presents a precise measure of distance in data spaces, potentially revolutionizing machine learning techniques like dimension reduction and clustering, using Riemannian metric manifold classes.

5. Literature Review

Without any complicated structure, several compact almost complex manifolds may be constructed in four dimensions. Finding a single higher dimensional manifold with nearly complicated structures is a challenging task.

Todd research characterized almost contact structures on G2-manifolds using co-symplectic objects properties, revealing that closed G2-manifolds admit almost contact metric 3-structure when constructing and characterizing it (Todd,2015). Borman studied over-twisted contact structures in all dimensions, focusing on extensions in dimensions > 3. He proved that a contact structure is acknowledged with an almost contact structure, studying a structure's geometric properties can be achieved via epimorphic geometry (Borman et.al.2014).

Puhle study on almost contact metric 5D manifolds found a metric connection compatible with almost contact structures. The space of torsion tensor splits into ten U (2)-irreducible subspaces, allowing 2^{10} classes of almost contact metric structures in 5-D manifolds. The study considered the normality property, namely $N\varphi + 2d\eta \otimes \xi = 0$, which is the integrable property (Mehmet,2005).

Gromove demonstrated that contact structures on an open manifold survey in h-principle, despite the questionable existence of contact structures on closed odd dimensional manifolds. An almost contact structure makes a contact structure and he proved that co-oriented contact structures in almost contact structures has a weak homotopy property (Gromove,1969). Sasaki introduced Sasaki Geometry, a geometric structure related to almost contact structures, which has been extensively studied and contributed to fundamental geometric relationships among manifolds (Sasaki,1960). Gray introduced almost contact manifolds as odd-dimensional manifolds with tangent bundle structure group reducible to U (n) × 1 (Gray,1959). In 1970, Kuo discovered a metric compatible with almost contact manifolds, demonstrating superior findings compared to classical perspectives on contact structures (Kuo,1070).

It is important to note from the above listed literature that some Geometers have already studied almost compact 1,2,3-manifolds to some extent. On any odd dimensional manifold M^{5n+3}, however, the existence of a nearly contact 4-structure is not well recognized. Specifically, given two almost contact structures. A third virtually contact structure on M^{4n+3} was presented by Kuo (1970), however the structure's validity was not established by demonstrating the tensor features of the structure. Furthermore, Tachibana and Yu postulated that there isn't a 4- almost contact structure that satisfies the anti-commutativity criterion on any odd-dimensional manifold (Tachibana &Yu,1970).

The 3- almost contact structure is proven to exist by this study, and it also creates a 4- almost contact structure (ϕ_4,ξ_4,η_4) on $M^{5n+4} \cong (N^{4n+3} \otimes R^d)$; $d|(2n+1)$ and g c d $(2, n) = 1$.

6. Preliminary Definition

6.1 Manifold

A manifold's atlas is a set of charts whose scope includes the manifold.

6.2 Fibre bundle

It is a manifold that, while not always a product globally, seems locally to be a product of two manifolds.

6.3 Topological Manifold

A manifold with topology A smooth manifold is defined as M with a smooth structure.

6.4 Riemannian Manifold

A Riemannian manifold is symmetric, positive definite, non-singular, 2-covariant tensor field.

6.5 Diffeomorphism

If f is both bijective and smooth in its inverse, then f is referred to as a diffeomorphism. This smooth map f is defined as $f : M \to N$.

6.6Almost Contact Manifold

Let M is a (2n + 1)-dimensional differentiable manifold and (φ, ξ, η) is a field of endomorphisms are a both vector field and a 1-form on M respectively. If the (φ, ξ, η) satisfies the three given situations

$$\phi^2 (Y_i) = -(Y_i) + \eta (Y_i)\xi \quad (2)$$

$$\eta(\xi) = 1 \quad (3)$$

$$\eta \circ \phi = 0 \quad (4)$$

then the triple (2), (3) and (4) are an almost contact structure where M is an almost contact manifold for $Y_i \in \Gamma (TM)$, $i \in N$ and non-singular vector ξ (Todd,2015).

6.7Almost contact 3-structure

Let M be an n-dimensional differential manifold, and let f, U, and u are a vector field, a 1-form, and a tensor field of type (1,1) respectively. If this tensor field satisfy

$$f^2 = -I + u \otimes U, \; f U = 0, \quad u \circ f = 0, \; u(u) = 1, \quad (5)$$

where, $u \circ f)(x) = u(fx)$ and I is the identity of this tensor field and (f, U, u) is termed an almost contact structure [13].

7. Methods and Discussion

7.1 Methodology

Defining the Almost Contact Structure: We begin by defining the tensor field ϕ_4, the vector field ξ_4, and the one-form η_4 on $(\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d)$ in a manner consistent with the properties of an almost contact structure.

Verification of Compatibility Conditions: We verify that the defined triple (ϕ_4, ξ_4, η_4) satisfies the compatibility conditions required for an almost contact structure.

Isomorphism with $\mathbb{M}^{(5n+4)}$: Through appropriate mappings and transformations, we establish the isomorphism between $(\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d)$ and the manifold $\mathbb{M}^{(5n+4)}$.

Extension: Leveraging the isomorphism, we extend the almost contact structure from $(\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d)$ to $\mathbb{M}^{(5n+4)}$ while preserving the essential geometric properties. Using a quick combinatorial analysis, we identified a limited number of possibilities resulting in a 4- almost contact structure for each of the 3- almost contact structures (ϕ_i,ξ_i,η_i) on $N^{(4n+3)}$ .This will be done by using algorithms. An almost complex structure and an almost contact structure have several characteristics.

As a result, each almost contact structure has a corresponding almost complex structure, and Ji's almost complex structures may be used to create them. Assume that for any i = 1, 2, 3, a differentiable manifold permits almost contact 3-structure (ϕ_i,ξ_i,η_i) for i = 1,2,3 on $N^{(4n+3)}$ of all permutations of (1,2,3) gives 3-almost complex structures J_1, J_2, J_3 linked with each almost contact structure(Kuo,1970). The 4-almost contact structure will be built on top of this base.

7.2 The Creation of the 4- structure (ϕ_4,ξ_4,η_4) on manifold $M^{(5n+4)} \cong (N^{(4n+3)} \otimes R^d$

For a 3-structure to be feasible, the following outcome is required.

Theorem 1. Let $\emptyset_1, \emptyset_2 \in T_{(1,1)}$, $\xi_1,\xi_2 \in TM$ and $\eta_1,\eta_2 \in TM^*$. Suppose $(\emptyset_1,\xi_1,\eta_1)$ and

$(\emptyset_2,\xi_2,\eta_2)$ are both almost contact structures, then

$$\emptyset_1 \emptyset_2 + \emptyset_2 \emptyset_1 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1$$

$$\emptyset_1 \xi_2 + \emptyset_2 \xi_1 = 0$$

$$\eta_{(1)} \circ \emptyset_2 + \eta_{(2)} \circ \emptyset_1 = 0$$

$$\eta_1(\xi_2)=0$$

holds and is the condition for an additional structure.

Proof. Let set

$\emptyset_3 = \emptyset_1 \emptyset_2 - \eta_2 \otimes \xi_1 = -\phi_2 \phi_1 + \eta_1 \otimes \xi_{(2,)}$ $\eta_3 = \eta_1 0 \phi_1 = -\eta_2 0 \phi_1$

and $\xi_3 = \phi_1 \xi_{(2)} = -\phi_{(2)} \xi_1$ (6)

Now, showing (ϕ_3, ξ_3, η_3) describes a 3-almost contact structure. If $X \in TM$, then also $X \in M^{(4n+3)}$ is a smooth manifold (Tachibana & Yu,1970).

$\phi_3^2 = -X + \eta_3 \otimes \xi_3$ (7)

Now we have to to show that $\eta_3 (\xi_3) = 1$

$$\eta_3 \ o\ \phi_3 = 0 \quad (8)$$

The equation (8) for (ϕ_3, ξ_3, η_3) for explain an almost contact structure on $M^{(4n+3)}$

First and foremost, $\eta_3 (\xi_3) = 1$.

Let, $\eta_3 = -\eta_2 \ o\ \phi_1$

$\xi_3 = \phi_1 \xi_2$,

$\eta_3 (\xi_3) = -\eta_2 \ O\ \phi_1 (\xi_3) = -\eta_2 (\phi_1 (\xi_3))$

But,

$$\phi_1 (\xi_3) = \phi_1 (\phi_1 \xi_2) = \phi_1^2 (\xi_2). \quad (9)$$

Substituting $\phi_1^2 = -1 + \eta_1 \otimes \xi_1$ in equation (9) gives

$$\phi_1^2 (\xi_2) = -\xi_2 + \eta_1 (\xi_2) \xi_1 = -\xi_2 + 0 = -\xi_2 \quad (10)$$

$-\eta_2 (\phi_1 \xi_3) = -\eta_{(2)} (-\xi_{(2)}) \ \eta_2 \xi_2 = 1$

$\eta_{(3)} (\xi_{(3)}) = 1.$

Now, $\phi_3 \xi_3 = 0$

Suppose that $\xi_3 = -\phi_2 \xi_{(1,)}$

Then, we get $\phi_3 \xi_3 = \phi_3 (-\phi_2 \xi_1) = \phi_3 (-\phi_{(2)} (X) \xi_1)$ (11)

Substituting $\phi_3 = \phi_1 \phi_{(2-)} \eta_2 \otimes \xi_{(1,)}$ in equation (11) gives

$$\emptyset_3 \xi_3 = \phi_1 \phi_2 (X) - \eta_2 (X) \xi_1 (-\phi_2 (X) \xi_1) \quad (12)$$

On simplifying the equation (12) gives

$$= -\phi_1 (\phi_2^2 (X) \xi_1) - 0 = -\phi_1 (\phi_2^2 (\xi_1)) \quad (13)$$

But, there is also,

$$\begin{aligned} \phi_2^2 (\xi_1) &= -\xi_1 + \eta_2 \otimes \xi_1 \\ &= -\phi_1 [-\xi_1 + \eta_2 \otimes \xi_1] \\ &= -\phi_1 [-\xi_1 + \eta_2 (X) \xi_1] \\ &= -\phi_1 [-\xi_1 + \eta_2 (\xi_2) \xi_1] \ : X = \xi_2 \\ &= -\phi_1 [-\xi_1 + \xi_1] \\ &= 0. \end{aligned}$$

Also, let $X = \xi_1$, then we have,

$$\begin{aligned} &= -\phi_1 [-\xi_1 + \eta_2 (\xi_1) \xi_1] \\ &= -\phi_1 \xi_1 \\ &= 0. \end{aligned}$$

gives $\phi_3 \xi_3 = 0$.

Finally, we show that, $\eta_3 \ o\ \phi_3 = 0$

Now, $\eta_3 \ o\ \phi_3 (X) = \eta_3 (\phi_{(3)} (X))$

Let, $\phi_3=\emptyset_1\ \phi_2-\eta_2\ \xi_{(1.)}$ Then, for $\eta_3=\eta_1 \circ \phi_2=-\eta_2 \circ \phi_1$ and $X \epsilon TM$.

$$\eta_3\ (\phi_3\ (X)) = \eta_3\ (\phi_1\ \phi_2\ (X)-\eta_2\ (X)\ \xi_1\ )$$
$$=-\eta_2 \circ \phi_1\ (\phi_1\ \phi_2\ (X)-\eta_2\ (X)\ \xi_1\ )$$
$$=-\eta_2 \circ \phi_1\ (\phi_1\ (\phi_2 (X))+\eta_2\ (X)\ \xi_1\ (\eta_2 \circ \phi_1)$$
$$=-\eta_2\ (\phi_1^2\ (\phi_2\ (X))+0$$
$$=-\eta_2\ [-\phi_2\ (X)-\eta_1\ (\phi_2\ (X))\ \xi_1]$$
$$=-\eta_2\ \phi_2\ (X)-\eta_1\ (\phi_2\ (X))\ \eta_2\ (\xi_1)$$
$$=-\eta_2 \circ \phi_2\ (X)-\eta_1\ (\phi_2\ (X)).0$$
$$=0-0=0$$

The result $\eta_3 \circ \phi_3=0$ explain that any two manifold – structures(ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2) represent same.

Theorem 2. Let M^d is a manifold as d divide 2n+1 and two types of coordinates (local and global) ($x_1, \cdots x_n, y_1, \cdots y_{(n,)}\ f$) with respect to

$\eta_4 = df_i - \sum_{(i=1)}^{4} y_i\, dx_i$ in M^d.

Proof. Let $U \subset M^d$. Let U be an open ball transverse to ξ_i that symmetrically simplistic in U.

The global coordinates ($x_1, \cdots x_n, y_1, \cdots y_{(n,)}\ f$) that $d\eta_4=\sum dx_i \wedge dy_i : i=1,\dots,4.$ (14)

$d(\eta_4+\sum_{(i=1)}^{4} y_i\, dx_i)=0$ so that $\sum_{(i=1)}^{4} y_i\, dx_i = df_4$ for some functions f_4.

Obviously, $\eta \wedge (d\eta)^n = df_4 \wedge dx_1 \wedge \dots \wedge dx_n \wedge \dots \wedge dy_n \neq 0$. (15)

Henceforth, the volume form does not vanish in U. So df_4 is independent of ($dx_i\ dy_i$)and thus we consider x_i, y_i and f_4 as a coordinate system. According to Tachibana and Yu [6]. we can construct a new structure ($\varphi 4$, $\xi 4$, $\eta 4$) from almost contact 3-structure such that

$$\eta_{(i)}\ (\xi_{(3)}) \neq \eta_{(4)}\ (\xi_{(i)}) \neq 0 \text{ for } i = 1,2,3.$$

The dimension of the manifold with the 4-almost contact structures (ϕ_1,ξ_1,η_1), (ϕ_2,ξ_2,η_2) , (ϕ_3,ξ_3,η_3), (ϕ_4,ξ_4,η_4) get the form 5n+4:gcd(2,n)=1. The theorem 2 demonstrates the extension of almost contact structure to more than four structures.

8. Novelty, Innovation, and Advancement

The extension of almost contact structures onto spaces of the form ($\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d$) and its isomorphism with the manifold $\mathbb{M}^{(5n+4)}$ represent a significant advancement in the field of differential geometry and geometric analysis. This research brings several novel contributions and innovations, pushing the boundaries of understanding in this area. Below are some key aspects highlighting the novelty, innovation, and advancement of this research:

Exploration of Unconventional Spaces: The extension of almost contact structures onto spaces formed by the tensor product of natural numbers and Euclidean spaces ($\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d$) is a departure from conventional settings. This exploration opens up new avenues for studying geometric structures in unconventional mathematical spaces.

Integration of Algebraic and Geometric Structures: By considering the tensor product of natural numbers with Euclidean spaces, this research bridges algebraic and geometric structures. The isomorphism with the manifold $\mathbb{M}^{(5n+4)}$ facilitates the integration of algebraic properties of tensor products with the geometric properties of manifolds, enriching the understanding of both domains.

Generalization to Higher Dimensions: The extension onto ($\mathbb{N}^{(4n+3)} \otimes \mathbb{R}^d$) and its isomorphism with $\mathbb{M}^{(5n+4)}$ provide a generalized framework applicable to higher-dimensional spaces. This generalization allows for the exploration of almost contact structures in higher dimensions, which has implications for various areas of mathematics and physics where higher-dimensional spaces are encountered.

Development of New Techniques: The extension of almost contact structures onto unconventional spaces requires the development of new mathematical techniques and methodologies. This research fosters innovation in mathematical analysis, including novel approaches to defining tensor fields, verifying compatibility conditions, and establishing isomorphisms between spaces.

Potential Applications in Physics: The advancement in understanding almost contact structures on ($\mathbb{N}^{(4n+3)}$

$\otimes \mathbb{R}^d$) and its isomorphism with $\mathbb{M}^{(5n+4)}$ holds promise for applications in theoretical physics. The geometric properties elucidated through this research may find applications in areas such as quantum field theory, string theory, and mathematical physics, contributing to the advancement of fundamental theoretical frameworks.

Interdisciplinary Connections: The interdisciplinary nature of this research, combining elements of differential geometry, algebra, and mathematical physics, fosters connections between diverse fields of study. This interdisciplinary approach encourages collaboration and cross-fertilization of ideas, leading to further advancements in each respective field.

Conclusion

This article introduces a class of almost contact structures, g c d (2, n) = 1, related to an almost compact 3-structure on a smooth Riemannian metric manifold. It demonstrates the existence of an almost contact structure (ϕ_4,ξ_4,η_4) on $(N^{(4n+3)}\otimes R^d)\cong M^{(5n+4)}$ from (ϕ_i,ξ_i,η_i) for i = 1,2,3 on $N^{(4n+3)}$,

References

Geiges, H. (1993), *Contact Structures on (n-1)-connected (2n+1)-manifolds,* Pacific J. Math.,161 (1),129-137.

Boothy W.M., Wang H.C. (1958), *On contact manifolds* Annals of Math.,68,721-734.

Borman M., Eliashberg Y., Murphy. E. (2014), *Existence and classification of over- twisted contact structures in all dimensions* arXiv.1404.6157v2 [math.SG].

Adara M.B. (2009), *An isoparametric function on almost k-contact manifold* An.St. Univ. Ovidus Constanta. 17(1),1522.

Mihai A. & Rosca R. (2005), *On vertical skew symmetric almost contact 3-structures* J. Geom.,82,146155.

Tachibana S., Yu, W. N. (1970). On a Riemannian space admitting more than one Sasakian structures. Tohoku Math.J.22,536-540.

Kuo, Y.Y. (1970), *On almost contact 3-structures* Tohoku Math.J.22,235-332.

Mehmet, A. (2005), *Submanifolds of Riemannian product manifold* Turk J. Math. 389-401.

Todd, A.J. (2015), *On Almost Contact Structure on G2-manifolds.* arXiv:1501.06966v1 [math. DG].

Gromove, M.L. (1969), *Stable mapping of foliation into manifolds* Izv. Akad. Nauk SSSR ser. Mat.,33:707-734.

Sasaki, S. (1960), *On differential manifolds with certain structure which are closely related to almost contact structures,* Tohoku Math J., 12,459-476.

Gray, J.W. (1959), *Some global properties of contact structures* Ann. of Math., 69, 2,421-45.

Adara, M.B., Feyza, E.E., Selcen Y.P. and Bilal E.A. (2018), *On the geometry*