# Graduate Level Students' Techniques and Difficulties in Proving Theorems of Abstract Algebra

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***Abstract***

*This paper has discussed different techniques of proving theorems of abstract algebra adopted by the students of graduate level followed by their difficulties revealed over there. For, three graduate students from the classroom of Master's degree level in mathematics education of Tribhuvan University were selected by using purposive sampling technique. The difficulties as experienced by students were explored through interviews with the help of interview guidelines; and their responses were recorded by using mobile phone. These recorded responses were transcribed and analyzed by using general inductive approach. The findings reveal that students have felt more difficulty in the indirect approaches of proofs in comparison to direct approaches while learning theorems in abstract algebra. The major difficulties as they experienced are in the selection of appropriate techniques of proofs, connection of previous concepts for logical arguments in proofs and construction of examples and counter examples of the concepts related to theorems. These difficulties are expected to be reduced if the teacher of abstract algebra course in higher mathematics education focuses on conceptual understanding and critical thinking for their students' learning.*

**Keywords**: Abstract algebra, direct proofs, indirect proofs, axiomatic approach, induction approach, students' difficulties

## Introduction

Abstract algebra is one of the mathematics courses in higher education which is more theoretical in nature. Algebra is the subject area of mathematics that studies algebraic structures such as groups, rings, fields, modules, vector spaces, and algebras (Ernst, 2016, p. 4). These broad algebraic structures include several definitions, examples, counter examples, proofs of statements, and other problem exercises related to concepts. The students who are learning abstract algebra courses in higher education are expected to have the capabilities of memorizing facts, definitions, and logic; using known facts and principles into new situation; constructing examples and counter examples of each abstract structure, and proving algebraic statements (Citation).

However, as Fry and others (2009) pointed out, "learning is not a single thing; it may involve mastering abstract principles, understanding proofs, remembering factual information, acquiring methods, techniques and approaches, recognition, reasoning, debating ideas, or developing behavior appropriate to specific situation; it is about change" (p. 8). These explanations are emphasizing that learning is the whole cognitive process and involves behavioral skills for graduate students to get mastery in abstract principles, facts and axioms in abstract algebra. This definition of learning equally reflects the learning of abstract algebra in higher education where students need to learn proofs of abstract algebraic statements.

Proving statements (theorems), lemma, propositions and corollaries accurately is an important skill required for the graduate students to learn abstract algebra where theorems are considered tools that make new and productive applications of mathematics possible (Judson and Beezer, 2015). In advanced mathematics courses in universities all over the world, students' construction and understanding mathematical proof is emphasized (Guler, 2016,). Thus, proving theorems are fundamental tasks to learn concepts in abstract algebra. Proof has many facets, for example, it is evidence in society, induction in science and deduction in mathematics (Nardi and Iannone, 2006); the importance of proof beyond a university degree is mentioned as: "it is eventually about using reason in everyday life" (Stefanowicz, 2014, p. 33).

But, how students are conceptualizing and experiencing proofs in abstract algebra at graduate level is the concern of this study. There are varieties of approaches of proofs that can be used to prove the theorems in abstract algebra. Lalonde (2013) stated four types of proofs of theorems in mathematics which are direct proofs, proof by contradiction, mathematical induction and proof by contrapositive. These techniques of proofs are also equally applicable in proving theorems in abstract algebra. That is why understanding proof techniques becomes essential for graduate students to learn the concepts in abstract algebra.

Moreover, it is expected that the students who are studying abstract algebra course in higher education need to have several cognitive and behavioral skills to prove the theorems. However, it is experienced that students have fraught of difficulties in proving theorems in this course. Algebraic arguments are highly valued by students but difficult to produce or understand (Nardi and Iannone, 2006); students have some common difficulties when learning new concepts because they have poor knowledge of mathematical quantifiers, lack of ability to select appropriate proof techniques, and inability to understand given hypothesis (Judson and Beezer, 2015). Also, students need to have different formulations of certain mathematical concepts to prove the mathematical statements (Lalonde, 2013).

Proving theorems in abstract algebra requires both conceptual and procedural knowledge, but improving the conceptual knowledge of students helps to increase the procedural knowledge in learning algebra (Booth and Koedinger, 2008). The pre-requisites like the concepts of numbers, ratios, proportions, order of operations, symbolism, equations, and functions are helpful to develop the conceptual knowledge in algebra, but lack of such pre-requisites among students even at higher level creates difficulties in learning (Welder, 2006). Thus, the students having sufficient conceptual knowledge in abstract algebra are expected to have fewer difficulties in selecting appropriate techniques of proving theorems.

Using examples in proving theorems of abstract algebra are expected to help the development of abstract concepts in higher mathematics learning (Judson and Beezer, 2015). They argued that using examples means giving insight into existing theorems and fostering intuitions as to what new theorem might be true; and they further stated that applications, examples, and proofs are tightly interconnected. Explaining examples, non-examples either alone or in combination of them is beneficial to improve conceptual understanding (Booth and others, 2013). Hence, how graduate students were conceptualizing and learning proof techniques and what difficulties they were encountering and experiencing while proving the theorems in abstract algebra were intended to document.

## Objectives of the Study

This research intended to explain the techniques of proofs that are implemented in proving theorems and to explore the difficulties experienced by graduate students while proving theorems in abstract algebra.

## Review of Relevant Literature

Mathematical proof is absolute, which means that once a theorem is proved it is proved forever (Stefanowicz, 2014). Proof is nothing more than a convincing argument about the accuracy of a statement (Judson and Beezer, 2015), as a sequence of logical statements, one implying another, which gives an explanation of why a given statement is true (Stefanowicz, 2014). Moreover, theorem is a justified assertion that some statement of the form $P \rightarrow Q$ is true and a proof is the argument that justifies the truth of the theorem where $P$ is given and $Q$ needs to be proved (Donaldson and Pantano, 2015). There are little differences among the varieties of algebraic statements all of which require proof. Judson and Beezer (2015, (p.2) have beautifully explained their distinction as:

*If we can prove a statement true, then the statement is called a proposition. A proposition of major importance is called a theorem. Sometimes instead of proving a theorem or proposition all at once, we break the proof down into modules; that is, we prove several supporting propositions, which are called lemmas, and use results of these propositions to prove the main result. If we can prove a proposition or a theorem, we will often, with very little effort, be able to derive other related propositions called corollaries.*

This description shows that there is logical difference between propositions, theorems, lemmas and corollaries in abstract algebra. However, these statements are similar in the sense of proofs because each of them needs to be justified by logical arguments.

But, the selected students are in confusion to explain these differences. One student stated as: "*in my opinion, theorems, proposition, lemmas and corollaries are the same thing. I cannot describe the difference of them. All of them require proofs which are equally important for me*". Similarly, next student opined as "*I heard the differences between theorems, corollaries, lemmas and proposition, but became confused to describe their distinction*".

These experiences of students indicate that students, even at graduate level, have great doubt to make distinction among theorems, propositions, lemmas and corollaries. However, all the selected students agreed that all of those statements require proofs in abstract algebra.

Lalonde (2013) stated four types of proof techniques in mathematics: direct proofs, proof by contrapositive (or indirect proofs), mathematical induction and proof by contradiction (pp. 137- 148). The direct method is more logically straightforward (Donaldson and Pantano, 2015) where we start with the hypothesis and make a chain of logical deduction to eventually prove the given proof (Lalonde, 2013). Also, contrapositive, and the contradiction arguments are quicker and more self-contained, but they require a deeper familiarity with logic (Donaldson and Pantano, 2015, p. 22).

***Direct proof***

It is a more popular and frequently used technique of proofs in abstract algebra. It assumes a given hypothesis or any known statement, and then logically deduces conclusion (Stefanowicz, 2014). To prove the theorems in the form if $P$ then $Q$, we can use this technique of proofs. According to Donaldson and Pantano (2015), we assume $P$ and logically deduce $Q$ in direct method of proof (p. 20). Likewise, argument is constructed using a series of simple statements, where each one should follow directly from previous one (Stefanowicz, 2014). Here, we follow hypothesis by supporting other true statements without missing any steps or gap in reasoning, but can use axioms or previously established theorems (ibid.). The following example displays the use of this technique in graduate abstract algebra.

<u>Consider a theorem</u>: *Every cyclic group is abelian*. Here, we may assume $G$ is a cyclic group and prove $G$ is cyclic (??) logically with the sequence of supporting true statements where one implies another from starting to ending. Since $G$ is cyclic group, then there exists an element $x$ in $G$ such that every element of $G$ can be expressed as some integer power of $x$ which is due to by definition of cyclic group. Let $a$ and $b$ be any two elements of $G$ then there exists integers $m$ and $n$ such that $a = x^m$ and $b = x^n$. Hence $a * b = x^m * x^n = x^{m+n} = x^{n+m}$ (the set of integers is commutative under addition) $= x^n * x^m = b * a$ where $*$ is the binary operation in $G$. Hence, $G$ is an abelian group.

The above proof in abstract algebra indicates that the hypothesis given in theorems is supported by series of valid arguments. These arguments are the definitions and already established facts. This direct method of proof is also called the formal deduction of proof in mathematics. The valid logics from starting to ending are justified by several axioms one after another.

### *Indirect proof*

This technique assumes the hypothesis together with the negation of a conclusion to reach the contradictory statement. It is often equivalent to proof by contrapositive, though it is subtly different (Stefanowicz, 2014, p. 11). Regarding the proof by contradiction technique, Donaldson and Pantano (2015) stated that for proving the theorem in the form if $P$ then $Q$, we assume that $P$ and not $Q$ is true and deduce the contradiction on $P$ (p. 20). The contradiction on $P$ implies that our assumption is wrong and thus the conclusion of the theorem is true.

If we wish to prove *the product of two primitive polynomials is primitive* (Gauss lemma, Hungerford, 1974, p. 162) then we have to use indirect method of proof as follows. Here, the hypothesis is not explicitly stated, but students need to be familiar with the definition of primitive polynomials. To prove this theorem, assume the product of two primitive polynomial $f(x) = a_0 + a_1x + \ldots + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ is not primitive where $f(x)\, g(x) = c_0 + c_1x + c_2x^2 + \ldots + c_{m+n}\, x^{m+n}$, with $c_k = a_kb_0 + a_{k-1}\, b_1 + \ldots + a_0\, b_k$ for $k = 0, 1, \ldots, m + n$. Then there exists a prime $p$ such that $p \mid c_k$ for all $k$. Since $c(f)$ is a unit then $p \nmid c(f)$, whence there is a least integer $s$ such that $p \mid a_i$ for $i < s$ and $p \nmid a_s$. Similarly, there is a least integer $t$ such that $p \mid b_j$ for $j < t$ and $p \nmid b_t$. Since $p \mid c_{s+t} = a_{s+t}b_0 + \ldots + a_{s+1}b_{t-1} + a_sb_t + a_{s-1}\, b_{t+1} + \ldots + a_0b_{s+t}$. Then $p$ must divide $a_sb_t$. Since $p$ is a prime element in the ring $R$, then $p$ must divide either $a_s$ or $p$ divide $b_t$ both of which are the contradiction. Hence our assumption is wrong which proves the product of $f(x)$ and $g(x)$ must be primitive.

This example justifies that indirect proof also follows the sequence of logical and convincing arguments but its starting point is different from the direct method of proofs.

### *Axiomatic approach of proof*

Axiomatic approach is one of the familiar approaches in proving theorems in geometry, which is equally valuable in proving theorems in abstract algebra. This approach of proving is based on the axioms of certain domain of mathematics. Morash (1987) stated that students are introduced to mathematics as a deductive science through plane geometry where we began with the set of axioms to prove the theorems, where theorems are deduced by means of proof with series of statements whereby their validity is based on an axiom or previously proved theorem (p. 149). Judson and Beezer (2015) explained:

In axiomatic approach of proof, we take a collection of objects S with definitions and assume some rules, called axioms, about their structures; and using these axioms (requiring consistent) for S we wish to derive other information about S by using logical arguments (p.1).

If we wish to prove *"if $x + y = x + z$ then $y = z$ for all $x, y, z$ belongs to ring $R$"*, then we have the following procedures of axiomatic techniques. $x + y = y + z$ implies $(-x) + (x + y) = (-x) + (x + z)$ (by the existence of additive inverse in $R$) $\Rightarrow ((-x) + x) + y = ((-x) + x) + z$ (by associativity of addition) $\Rightarrow (x + (-x)) + y = (x + (-x)) + z$ (by the commutativity of addition) $\Rightarrow 0 + y = 0 + z$ (by existence of additive inverse) $\Rightarrow$ y = z (0 is additive identity).

This technique of proof also indicates that it is a direct method of proof in which each statement is supported by certain axioms and known results in ring theory. Based on such reasons we can reach the conclusion of the theorem. This technique is generally used to test the certain structural properties in justifying algebraic statements and to establish the truth of examples and counter examples in abstract algebra. There are several problems in abstract algebra which require such axiomatic approaches to proof as to prove group, ring, field etc.

### *Induction approach*

This is also a familiar approach to prove theorems in abstract algebra. This method of proof is referred to as principle of mathematical induction. In mathematical induction "we assume $p(n)$ be an infinite collection of statements with $n \in N$; prove $p(1)$ is true; assume the theorem is true for $n = k$, that is $p(k)$ is true; and prove $p(k + 1)$ is also true for all $k$ in $N$, then conclude $p(n)$ is true for all $n$ in $N$ (Stefanowicz, 2014, p. 20).

This method of proof can be used to prove Sylow's first theorem in group theory which states: *"Let $G$ be a group of order $p^n m$, with $n \geq 1$, $p$ prime, and $(p, m) = 1$. Then $G$ contains a subgroup of order $p^i$ for each $1 \leq i \leq n$ and every subgroup of $G$ of order $p^i$ $(i < n)$ is normal in some subgroup of order $p^{i+1}$"* (Hungerford, 1974).

For $n = 1$, we have $|G| = pm$ and $p$ is a prime then by Cauchy's theorem $G$ contains an element '$a$' of order $p$, and therefore a subgroup $<a>$ of order $p$.

Hence the theorem is true for $n = 1$. Now assume $n > 1$ and $H$ is a subgroup of $G$ of order $p^i$ ($1 \leq i < n$), then $[G: H] = (|G|) / (|H|) = (p^n m)/(p^i) = p^{n-i}m$ implies $p$ / $[G: H]$ and $H$ is normal subgroup of $G$, then $N_G(H) \neq H$. Also $1 < |N_G(H)/ H| = [N_G(H): H] \equiv [G: H] \equiv 0(\text{mod } p)$. Hence $p$ divides $|N_G(H)/ H|$ and so $N_G(H)/ H$ contains a subgroup of order $p$ by Cauchy theorem. Since the subgroup of $N_G(H)/ H$ is of the form $H_1/ H$ where $H_1$ is the subgroup of $G$ containing $H$. Since $H$ is normal subgroup of $N_G(H)$, $H$ is necessarily normal in $H_1$. Finally, $|H_1| = |H|/(|H_1/H|) = p^i p = p^{i+1}$. Thus $H_1$ is a subgroup of $G$ of order $p^{i+1}$. Hence the theorem is true for $n = i + 1$ if it is true for $n = i$. So by induction theorem is true for every $n$.

Here, induction is used in combination with the formal deduction approach. There are several arguments as in direct and indirect proofs together with the process of induction. However, the major approach is mathematical induction in this proof.

### *Contrapositive approach*

Here, we show the contrapositive statement is true and then conclude the given conditional is true. That is, we assume $\sim Q$ and deduce $\sim P$ to prove the statements in the form $P \rightarrow Q$ (Donaldson and Pantano, 2015, p. 20). This method is like a sub-method of contradiction, but the argument begins with $\sim Q$ and establishes $\sim P$. One example of using such approach is as follows.

*Theorem: If R is a unique factorization domain with quotient field F and f (x) is a primitive polynomial of positive degree in* $R[x]$*, then f (x) is irreducible in* $F$[x] *if f(x), is irreducible in* $R[x]$. *Proof:* Suppose $f(x)$ is not irreducible in $F[x]$ then $f(x) = g(x)h(x)$ with $g$ and $h$ having positive degrees. Then we can show $g = (a/b)\ g_2$ where $g_2 \in R[x]$ is primitive polynomial with $deg\ g = deg\ g_2$ and $a, b \in R$. Similarly, we get $h = (c/d)\ h_2$ where $h_2$ is primitive in $R[x]$ with $deg\ h = deg\ h_2$ and $c, d \in R$. Hence, we get $(bd)f = acg_2h_2$; and since $f$ and $g_2h_2$ are primitive, it implies $bd$ and $ac$ are associates in $R$. Thus, $f$ and $g_2h_2$ are associates in $R[x]$ which implies that $f(x)$ is not irreducible in $R[x]$, which is a contradiction. Hence, by contrapositive method we prove the theorem.

### *Proving theorem by counter example*

Using examples and counter examples in proving theorems in abstract algebra are like using concrete materials in teaching and learning school mathematics. Examples can help students to develop insights in proving approaches of the theorems. A theorem cannot be proved by examples; however, the standard way to show that a statement is not a theorem is to provide a counter example (Judson and Beezer, 2015, p. 3).

For example, *every prime ideal is not a maximal ideal in the commutative ring with identity* can be proved by providing counter example. That is, in the ring of integers Z, the zero ideal (0) is prime which is not maximal ideal because there are infinitely many ideals, in particular 2Z in between (0) and Z. This counter example justifies (0) is not maximal ideal in Z. Similarly, every nilpotent group is solvable but every solvable group is not nilpotent can be verified by showing the symmetric group $S_3$ is not nilpotent.

Finally, Donaldson and Pantano (2015) pointed out that the direct method has the advantage of being easy to follow logically. The contrapositive method has its advantage when it is difficult to work directly with proposition $P \rightarrow Q$, especially if one or both involve the non-existence of something. Hence, in proving theorems in abstract algebra we generally use the formal deduction approach. However, the use of other techniques of proofs including mathematical induction are equally applicable.

## Methodology

I believe in interpretivist research paradigm which considers relativist ontology, subjective epistemology and qualitative methodology (Guba and Lincoln, 2005). That is, reality is contextual for which both knower and known involve in the construction of new knowledge by applying the qualitative process of research. Under these philosophical assumptions, I used descriptive case study and applied inductive process of research (Gillham, 2000). I selected three case students purposively from the mathematics classroom of Master's level. The purpose of selection was to include the experiences of low level, average and above average students.

I prepared semi-structured interview guidelines on the basis of objectives and then conducted face to face in-depth interviews. I explained the purpose of my study to students before taking interviews and assured them the information is used only for research which helped them to express their experiences freely. I took interviews individually and recorded their explanations by using mobile phone. I analyzed the information and interpreted the result by using the general inductive approach as described by Thomas (2006).

## Results and Discussion

Writing proofs is the essence of mathematics studies, in particular of abstract algebra. Generally, at university level, there is dependence on lecture with explanation in which every word will be defined, notations are clearly presented and each theorem is proved (Stefanowicz, 2014, p. 10). However, understanding proofs in abstract algebra is a taxing job for students even at graduate level. This section describes what common difficulties are experienced by graduate students while learning proofs in abstract algebra.

The interviews conducted among the students revealed the fact that students have felt difficulties in selecting appropriate techniques of proof. Selecting a method of proof is often a matter of taste (Donaldson and Pantano, 2015). One student stated:

> *I am unable to select appropriate methods of proof even I know the hypothesis and conclusion of the theorem, always confusing on: where to start proof? Which method is suitable? Why is this method in the book? But coping teachers' idea and try to memorize techniques. If I need to prove new statements from the exercise, I cannot select suitable proof techniques.*

The other student expressed his experiences as: *I am enjoying learning proof techniques by direct methods but it is difficult to understand indirect methods of proof in abstract algebra.*

These responses indicate that students have felt difficulties in choosing appropriate techniques of proofs. Their experiences also reveal that direct proof techniques are easy for conceptual learning in abstract algebra. Similarly, third student stated,

> *I am always confusing on how to start proof of the theorem and where to start it, sometime proof starts from definition of known concepts, sometime by hypothesis like direct method and sometime indirect techniques such as in Gauss lemma, but choosing indirect methods of proof is more difficult for me.*

Likewise, other students experienced as: *We feel comfortable to use axiomatic approach in proving theorems but if we forget some axioms required for the proof then we stop there, cannot do anything further.* They further opined: *There is mixture of induction with axiomatic and formal* *deduction approach in proving theorems in abstract algebra which creates difficulty in selecting and proving theorems.*

These responses of graduate students show that they are enjoying in axiomatic and direct approaches of proofs, but are unable to explain which method they need to select and use in proving theorems which is due to the mixture of several approaches in proving the same theorems in abstract algebra.

Moreover, connecting previous concepts like definitions, theorems and axioms are necessary arguments in proving theorem of abstract algebra. For such things, graduate students have several experiences including the following.

*In each type of proofs in abstract algebra, we are unable to connect basic facts while proving theorem, due to which we are trying to memorize teachers' proofs together with definitions. Similarly, they further opined that we have several difficulties to connect previous axioms, facts and logic to prove theorem like: every cyclic group is abelian…for example we need to memorize definitions of cyclic group, abelian group, binary operation and commutativity property in the set of integers… how connect these thing to get conclusion are the difficult aspects.*

These responses display that students have felt difficulty in connecting previous concepts in logical arguments while proving theorems. They are just trying to memorize proofs provided by teachers rather than understand the meaning and arguments in the proofs.

Also, constructing examples and counter examples besides proving any theorem is an important learning skill for graduate students. One student opined: *If I know example then easy for conceptual and procedural understanding in proofs of theorem, but difficult to construct example and counter example*. Next student responded, *I am completely unable to construct counter examples of the algebraic concepts and theorems even I know the proof of the theorem.*

These views indicate that students have felt comfortable to learn theorem if they can construct the examples and counter examples of the theorem. But their experiences indicate that constructing examples and counter examples is very difficult for them while learning proofs in the theorems.

Judson and Beezer (2015) pointed out that students often make some common mistakes when they are first learning how to prove theorem and to use different methods of proofs. These common difficulties are due to lack of appropriately conceptualizing mathematical quantifier, trying to prove theorem only by examples, assuming hypothesis which is not explicitly stated in the theorem, and being unable to select appropriate method of proofs. That is why, in indirect techniques of proof, students need strong foundation of mathematical language like quantifier and strong conceptual understanding. According to Donaldson and Pantano (2015), indirect proofs require a deeper familiarity with logic, so due to lack of such concepts of logic students at graduate level have faced difficulties in proving theorem by indirect techniques. However, direct technique of proof is easiest one because it does not require knowledge of any special techniques, it is hard to find a starting point to the proof of theorems (Stefanowicz, 2014).

## Conclusion

Proving theorems is one of the major objectives of graduate students in abstract algebra. There are several techniques of proofs including direct methods and axiomatic approaches; indirect approaches like method of contradiction, method of contrapositive; and mathematical induction. There are fundamental differences in such proof techniques in abstract algebra, however, they have some common characteristics which consider proof as a logically justified argument where every statement in the argument is supported by previous one until reaching at the ending process. The experiences of graduate students have indicated that direct methods are easier than indirect methods of proofs. The common difficulties of graduate students while learning proofs in abstract algebra are emerging due to lack of capability in selection of appropriate techniques of proofs, being unable to connect previous concepts for the logical arguments in proofs and lack of capability in constructing examples and counter examples of the concepts related to theorem. These difficulties can be reduced if the teachers in higher education focus on conceptual understanding and critical thinking for their students' learning.

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