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# The Golden Ratio: A Mathematical and Aesthetic Marvel

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## Abstract

The Golden Ratio, approximately 1.618, is a mathematical constant that has captivated the imagination of scholars, artists, architects, and scientists for centuries. This article delves into the mathematical underpinnings of the Golden Ratio, tracing its origins and historical significance in ancient civilizations and its evolution in mathematical theory. Known for its unique properties, the Golden Ratio is often associated with beauty, balance, and harmony, which has led to its widespread application in diverse fields. From the design of iconic architectural marvels like the Parthenon to its presence in Renaissance art, the Golden Ratio continues to inspire creativity. It also manifests in nature, evident in the arrangement of leaves, the spiral patterns of shells, and the structure of galaxies. In modern science, the Golden Ratio finds relevance in engineering, computer algorithms, and even financial models. This exploration underscores its enduring allure, showcasing its theoretical elegance and practical utility across time and disciplines.
**Key words**: Golden ratio, Fibonacci number, Golden rectangle, Golden spiral, Fibonacci spiral.

## Introduction

The Golden Ratio, denoted by the Capital Greek letter Φ (phi), is an irrational number that appears in various aspects of mathematics, art, architecture, and nature. Defined algebraically as $(1 + \sqrt{5})/2$, the Golden Ratio has been studied since ancient times and is often associated with aesthetic beauty and harmony [Akhtaruzzaman & Shafie (2011),p:1-22]. The golden ratio is associated with the Fibonacci sequence in a very simple way. The sequence is an example of a quadratic recursive equation sometimes used to describe various scientific and natural phenomena such as age-structured population growth. This article provides a comprehensive overview of the Golden Ratio, tracing its historical roots, mathematical properties, and diverse applications. In the following sections, first the golden ratio is defined mathematically and introduces later from geometrical point of view. Its appearance in nature, modern technology and architecture are reviewed very briefly.

## Historical Background

The golden section constant has incomparably richer history dating from ancient times, from Egypt to ancient Greece. Parthenon, whose construction started in 447 BC, was designed in the golden section proportions [Dunlap (1997)]. The concept of the Golden Ratio dates back to ancient civilizations. The earliest known references to the ratio appear in Euclid's "Elements" around 300 BCE. Euclid described it as the division of a line into two segments such that the ratio of the whole line to the longer segment is the same as the ratio of the longer segment to the shorter one. That is, about 300 B.C., Euclid of Alexandria, the most prominent mathematician of antiquity, gathered and arranged 465 propositions into thirteen books, entitled The Elements, denote by [AB] and AB the closed line segment with endpoints A and B and its length, respectively.

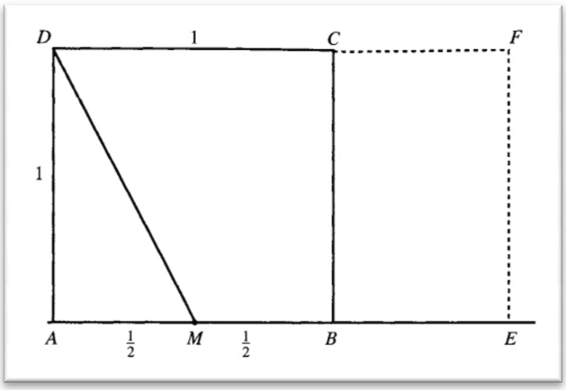



Figure 1. The mean proportion AE : AB :: AB : BE

In the Book VI, given a segment AE, find the point B for which AE/AB = AB/BE. We show his construction in Figure 1 in order to generalize it later. Start with the unit square ABCD and let M be the midpoint of AB. Construct the line segment MD. Draw a circle with center at M and radius MD so that it cuts A- at the point E. So, MD = ME. In modem notation, we have the lengths, ME = $\frac{\sqrt{5}}{2}$, MB = $\frac{1}{2}$, BE = $\frac{\sqrt{5}-1}{2}$.

Now, since AE = AB + BE, or AE = 1 + BE, we can write: AE = $\frac{1+\sqrt{5}}{2}$. Thus, we can easily show that AE/AB = AB/BE. The length AE, $\frac{1+\sqrt{5}}{2}$, is denoted by $\Phi$, and is called the golden ratio, or the divine proportion. In the above figure, the rectangle AEFD is called the golden rectangle. The history of the golden ratio pre-dates Euclid.

In modem notation, we have the lengths, known as the golden ratio has always existed in mathematics, it is unknown exactly when it was first discovered and applied by mankind. It is reasonable to assume that it has perhaps been discovered and rediscovered throughout history, which explains why it goes under several names, such as golden section, golden mean, golden number, divine proportion, divine section and golden proportion.

During the Renaissance, artists and architects such as Leonardo da Vinci and Luca Pacioli embraced the Golden Ratio, believing it to be a divine proportion that embodied perfection and beauty. Da Vinci's "Vitruvian Man" and Pacioli's treatise "De Divina Proportione" highlight the Golden Ratio's influence on art and design.

## Tools and Methods

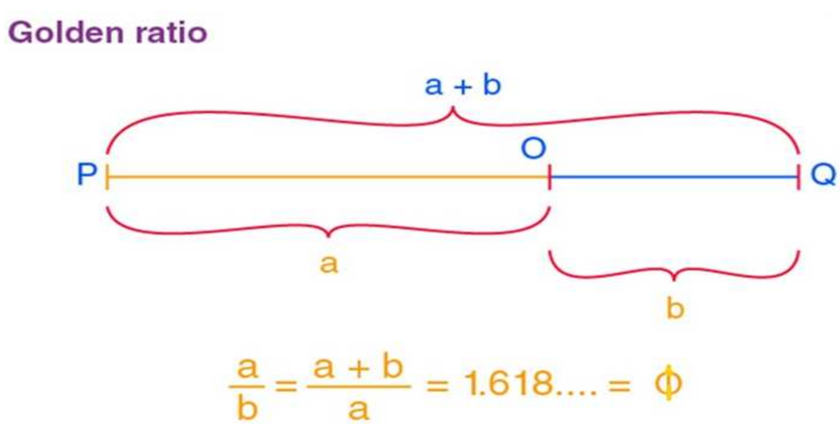
To explore the Golden Ratio, a combination of mathematical tools and geometric constructs were employed, illustrating its presence in both theoretical and practical contexts. Analytical methods such as algebraic derivations, geometric representations, and computational visualizations were used to validate the properties of the Golden Ratio. Historical tools like compass and straightedge were revisited to understand classical constructions, while modern software aided in visualizing complex patterns and relationships. The study utilized algebraic equations, Fibonacci sequences, and dynamic geometry software to demonstrate the ubiquitous nature of the Golden Ratio across various domains. To bridge historical and modern perspectives, tools ranging from simple diagrams to advanced computational algorithms were integrated into the analysis.

## Result and discussion

### Mathematical Properties of golden ratio

Two quantities a and b are said to be in the golden ratio Φ if

$$\frac{a+b}{a} = \frac{a}{b} = \Phi.$$




One method for finding the value of Φ is to start with the left fraction.

$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\Phi}$.

Therefore, $\Phi = 1 + \frac{1}{\Phi}$. …………………….(1)

This gives $\Phi^2 - \Phi - 1 = 0$ ....………………..(2)

Straightforward application of the quadratic formula results in $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. The negative of the negative root of the quadratic equation (2) is what we will call the golden ratio conjugate φ, (the small Greek letter phi), and is equal to $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$. The relationship between the golden ratio conjugate φ and the golden ratio Φ, is given by φ = Φ − 1, or using (1), $\varphi = \frac{1}{\Phi}$.

**Golden ratio in a continued fraction:**

One of the wonderful presentations of golden ratio on a continued fraction is as shown:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

This continued fraction produces a quadratic equation

$\Phi = 1 + \frac{1}{\Phi}$,

Or, $\Phi^2 - \Phi - 1 = 0$.

**Golden ratio in a continued square root:**

Another wonderful presentation of golden ratio on a continued square root is as shown:

$$\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

Or, $\Phi = \sqrt{1 + \Phi}$

Squaring both sides

Or, $\Phi^2 = 1 + \Phi$

Or, $\Phi^2 - \Phi - 1 = 0$

**Golden ratio in a Fibonacci sequence**

The golden ratio is associated with the Fibonacci sequence in a very simple way. The sequence is an example of a quadratic recursive equation sometimes used to describe various scientific and natural phenomena such as age-structured population growth. In order to define the general quadratic recursive formula, let $x_0$, $x_1$, p, and q be fixed positive numbers, and for any integer n > 2, define $x_n$, as

$\boldsymbol{x_n = px_{n-1} + qx_{n-2}}$ **……………………..(3)**

Murthy [8] provides a number of theorems for this general recursive equation. It is clear that many of the features that are proclaimed to be unique to the Fibonacci sequence are, indeed, common to all second-order recursive equations. For example,

$\boldsymbol{\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = r}$. **…………………(4)**

Where, r is the positive root of the quadratic equation

$\boldsymbol{x^2 - px - q = 0}$. **…………………(5)**

obtained by assuming that $x_n$, = x" is a solution to (1). One of our objectives in this paper is to show that if q = 1 and r is the limit in (2), then the pair (r, p) has all of the geometric and algebraic properties that are often ascribed as being unique to the pair (Φ,1) . For example, we have

$r - p = \frac{1}{r}$

Corresponding to the property, $\Phi - 1 = \frac{1}{\Phi}$.

Or, $$\Phi^2 - \Phi - 1 = 0$$

The recursion relation for the Fibonacci numbers is given by

$F_{n+1} = F_n + F_{n-1}$.

Dividing by $F_n$ yields

$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$………………..(6)

We assume that the ratio of two consecutive Fibonacci numbers approaches a limit as $n \to \infty$. Define limit $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$ so that $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\alpha}$ . Taking the limit, (6) becomes α = 1 + 1/α, the same identity satisfied by the golden ratio. Therefore, if the limit exists, the ratio of two consecutive Fibonacci numbers must approach the golden ratio for large n, that is,

$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$.

The ratio of consecutive Fibonacci numbers and this ratio minus the golden ratio is shown in Table 1. The last column appears to be approaching zero.

Table 1: Ratio of consecutive Fibonacci numbers approaches $\Phi$.

| n | $\frac{F_{n+1}}{F_n}$ | value | $\frac{F_{n+1}}{F_n} - \Phi$ |
|---|---|---|---|
| 1 | 1/1 | 1.0000 | -0.6180 |
| 2 | 2/1 | 2.0000 | 0.3820 |
| 3 | 3/2 | 1.5000 | -0.1180 |
| 4 | 5/3 | 1.6667 | 0.0486 |
| 5 | 8/3 | 1.6000 | -0.0180 |
| 6 | 13/8 | 1.6154 | -0.0070 |
| 7 | 21/13 | 1.654 | -0.0026 |
| 8 | 34/21 | 1.6190 | 0.0010 |
| 9 | 55/34 | 1.6176 | -0,0004 |
| 10 | 89/55 | 1.6182 | 0.0001 |

## Applications in Art and Architecture

### The golden rectangle

A golden rectangle is a rectangle whose side lengths are in the golden ratio. In a classical construction, first one draws a square. Second, one draws a line from the midpoint of one side to a corner of the opposite side. Third, one draws an arc from the corner to an extension of the side with the midpoint. Fourth, one completes the rectangle. The procedure is illustrated in Fig. 2.

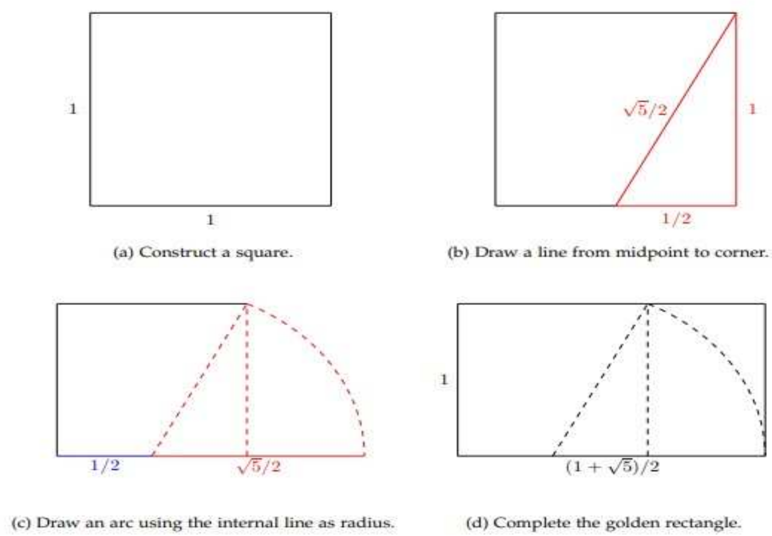



Figure 1.1. Classical construction of the golden rectangle

### The golden spiral

The celebrated golden spiral is a special case of the more general logarithmic spiral whose radius r is given by

$$r = ae^{b\theta} \text{ ……………………(7)}$$

where, θ is the usual polar angle, and a and b are constants. Jacob Bernoulli (1655-1705) studied this spiral in depth and gave it the name spira mirabilis, or miraculous spiral, asking that it be engraved on his tombstone with the enscription "Eadem mutata resurgo", roughly translated as "Although changed, I arise the same." A spiral was engraved at the bottom of his tombstone, but sadly it was not his beloved logarithmic spiral.

The golden spiral is a logarithmic spiral whose radius either increases or decreases by a factor of the golden ratio $\Phi$ with each one-quarter turn, that is, when θ increases by π/2. The golden spiral therefore satisfies the equation

$$r = a\Phi^{\frac{2\theta}{\pi}} \text{ …………….. (8)}$$

In our figure of the spiraling squares within the golden rectangle, the dimension of each succeeding square decreases by a factor of $\Phi$, with four squares composing each full turn of the spiral. It should then be possible to inscribe a golden spiral within our figure of spiraling squares. We place the central point of the spiral at the accumulation point of all the squares, and fit the parameter a so that the golden spiral passes through opposite corners of the squares. The resulting beautiful golden spiral is shown in Fig. 3.

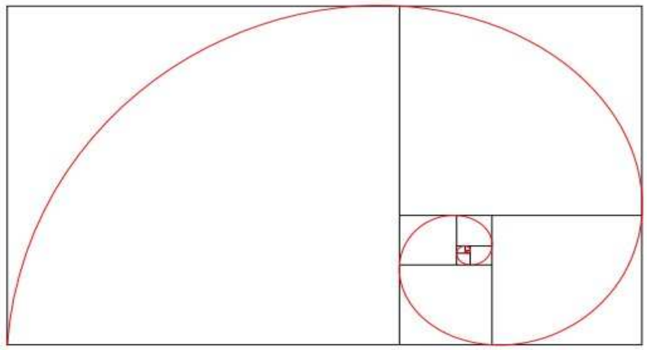


Figure 2.1. The golden spiral, the central point is where the squares accumulate

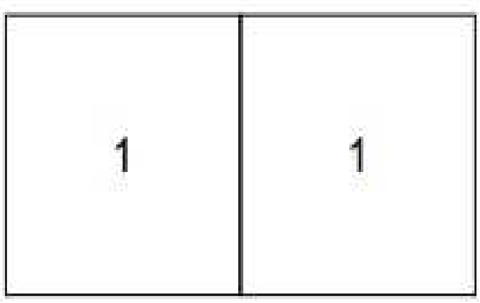
**The Fibonacci spiral**

Consider again the sum of the Fibonacci numbers squared:

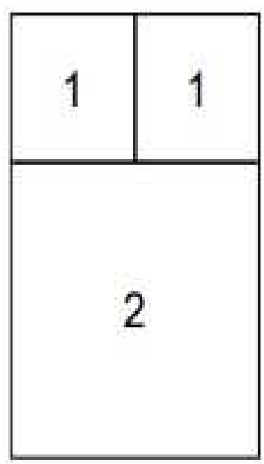
$\sum_i^n F_i^{\,2} = F_n F_{n+1}$ ………………………..(9)

This identity can be interpreted as an area formula. The left-hand-side is the total area of squares with sides given by the first n Fibonacci numbers; the right-hand-side is the area of a rectangle with sides $F_n$ and $F_{n+1}$.

For example, consider n = 2. The identity (8) states that the area of two unit squares is equal to the area of a rectangle constructed by placing the two unit squares side-by-side, as illustrated in Fig. (3.1.a.)









(a) n=2: $1^2 + 1^2 = 2$ (b) n=3: $1^2 + 1^2 + 2^2 = 6$

Figure 3.1. Illustrating the sum of the Fibonacci numbers squared. The center numbers represent the side lengths of the squares.

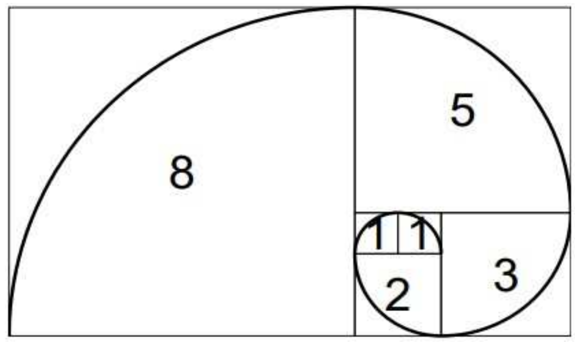



Figure 3.2. The sum of Fibonacci numbers squared for n = 6. The Fibonacci spiral is drawn

For n = 3, we can position another square of side length two directly underneath the first two unit squares. Now, the sum of the areas of the three squares is equal to the area of a 2-by-3 rectangle, as illustrated in Fig. (3.1.b.) The identity (9) for larger n is made self-evident by continuing to tile the plane with squares of side lengths given by consecutive Fibonacci numbers.

The most beautiful tiling occurs if we keep adding squares in a clockwise, or counterclockwise, fashion. Fig. (3.2) shows the iconic result obtained from squares using the first six Fibonacci numbers, where quarter circles are drawn within each square thereby reproducing the Fibonacci spiral.

**Self-similar decompositions of rectangles of any ratio and spirals:**

For any r > 1, and any rectangle R with a ratio of r to 1, we can divide R into two smaller rectangles, one of which has the same ratio r to 1, and this in turn can be so divided etc., as in Figure 2. The coordinates of the vertices $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$,… are given below.

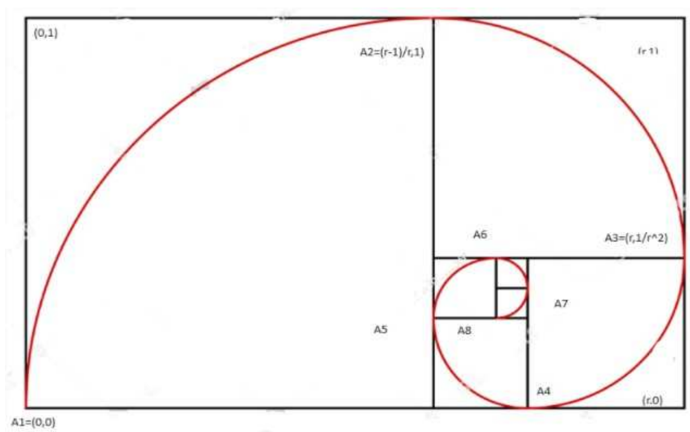



Figure.11. A rectangle of dimensions r to 1, with similar sub rectangles and a spiral

$A_1=(0,0)$, $A_2=(\frac{r-1}{r}, 1)$, $A_3=(r,\frac{1}{r^2})$,$A_4 = (r-\frac{1}{r}+\frac{1}{r^2}, 0)$, $A_5 = (r-\frac{1}{r}+\frac{1}{r^2}-\frac{1}{r^5}, \frac{1}{r^2})$

The vertices $A_1$, $A_2$, $A_3$, ... converge to a "center-point," $(\frac{r^3}{1+r^2}, \frac{1}{1+r^2})$.

**Spirals.** An equiangular spiral can be drawn through the points $A_1$, $A_2$, $A_3$, ... in Figure 2. Just use the center point $(\frac{r^3}{1+r^2}, \frac{1}{1+r^2})$. The central point is where the squares accumulate.

and fit a polar equation for the logarithmic spiral through any two of these points[5]. It is a general property of all such spirals that the tangents to the spiral at any point make a fixed angle with the rays from the center point[ Markowsky,(1992),P:2-19].

Artists and architects have long utilized the Golden Ratio to create visually pleasing compositions. In painting, the Golden Ratio is often employed to structure the layout and proportions of the canvas. Notable examples include works by Leonardo da Vinci, Salvador Dalí, and Piet Mondrian.

In architecture, the Parthenon in Athens and the pyramids of Egypt are believed to incorporate the Golden Ratio in their design. Modern architecture also embraces this ratio; the United Nations Headquarters and the Guggenheim Museum are examples of buildings influenced by φ.

**Golden ratio in nature**

The Golden Ratio appears in various natural phenomena, reflecting its fundamental role in the growth and structure of living organisms. Examples include:

a. **Phyllotaxis**: The arrangement of leaves, seeds, and flowers often follows a spiral pattern governed by the Golden Ratio, optimizing exposure to sunlight and space efficiency.
b. **Animal Morphology**: The proportions of certain animal bodies, such as the spirals of shells and the branching of trees, exhibit the Golden Ratio.
c. **Human Anatomy**: The human body, including the proportions of the face and limbs, can be analyzed through the lens of the Golden Ratio, contributing to perceptions of beauty and symmetry.
d. **Leonardo Da Vinci**: Many artists who lived after Phidias have used this proportion. Leonardo Da Vinci called it the 'Divine Proportion' and feathers it in many of his paintings for example in the famous "Mona Lisa"
e. **The vitruvian Man:** Leonardo did an entire exploration of the human body and the ratios of the lengths of various body parts. The vituvian man illustrates that the human body is proportioned according to golden ratios.
f. **The Parthenon:** The exterior dimensions of the in Parthenon in Athens, built in about 440BC, form a perfect rectangles.
g. **Egyptian pyramids:** The base lengths of pyramids divided by height of them gives a golden ratio.
h. **Sea shells:** The shape of inner and outer surfaces of the sea shells.

i. **Petals of Flowers:** Many flowers have a number of petals that is a Fibonacci number. The arrangement of petals often exhibits the Golden Ratio, optimizing exposure to sunlight and space.

j. **Galaxies:** Spiral galaxies, such as the Milky Way, follow a logarithmic spiral pattern, which is closely related to the Golden Ratio.

k. **Hurricanes:** Hurricanes display a spiral pattern similar to that of galaxies and shells, adhering to the logarithmic spiral that approximates the Golden Ratio.

l. **Photography:** Photographers use the Golden Ratio to compose images that are aesthetically pleasing. The Rule of Thirds is a simplified version of this principle, dividing an image into sections that are pleasing to the eye.

m. **Spider-Webs:** Some spiders build webs with spiral patterns that follow the Golden Ratio, optimizing the structural efficiency and strength of the web.

n. **Pinecones:** The scales of pinecones are arranged in a spiral pattern, with the number of spirals typically corresponding to Fibonacci numbers, demonstrating the Golden Ratio.

### Modern Scientific Applications

In contemporary science and technology, the Golden Ratio continues to find relevance. It appears in the analysis of financial markets, where certain trading algorithms utilize $\Phi$ for predictive modeling. Additionally, the ratio is explored in computer graphics and design, where it aids in creating harmonious and visually appealing interfaces.

Like the golden ratio in art and architecture, this balance leads to more elegant and effective AI solutions. Achieving the golden ratio of AI requires both technical expertise and an understanding of human needs and preferences. When AI systems are designed with the golden ratio in mind, they are more likely to provide meaningful and useful insights to users. Overall, the golden ratio of AI represents an exciting opportunity to create smarter, more human-centered technology solutions [Thapa & Thapa (2018), P:188-199].

## Conclusion

The Golden Ratio remains a captivating and multifaceted concept that bridges mathematics, art, nature, and modern science. Its unique properties and widespread occurrences underscore its significance and enduring appeal. As research continues, the Golden Ratio is likely to reveal even more about the intrinsic harmony and structure of the world around us.

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