DOI: https://doi.org/10.3126/cognition.v6i1.64458

A Study of Geodesic Equation From Variational Principle

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Abstract

This paper employs variational principle in studying geodesic Eq^{n}. In mathematical studies, a variational principle enables a problem to be solved employing calculus of variations that concerns seeking functions that increase the values of quantities that rely on those functions. For instance, the problem of ascertaining the shape of a hanging chain suspended at both ends can be solved using variational calculus. Hence, the variational principle is a function that lessens the gravitational potential energy of the chain. Geodesic Eq^{n} is a procedure used in mathematics, specifically in Riemannian geometry that results in obtaining geodesics. Actually, these represent the paths of particles with no proper acceleration, their motion pleasing the geodesic equations. As the particles are subject to no appropriate acceleration, the geodesics generally signify the straightest path between two points in a bent spacetime. The article has investigated into the relation between the variation principle and geodesic equations that are particularly used in general relativity in physics as well.

Key Words: Geodesic, equation, relativity, geometry, acceleration, variational principle

Introduction

This paper demonstrates the relationship between variational principle and studying geodesic Eq^{n}. In the regard of variational principle, Adhikari (1998) in his PhD dissertation states that every numerical calculation is finally inexact. Therefore, the efficient methods for the constructin of trustworthy approximations are practically significant. Variational principles permit the readers to use a proper solution of the deep equation to calculate the necessary quantity with fewer errors. Hence, almost all practical calculations use any one of the existing devices. Similarly, Cassel (2013) has viewed that there is a revival of application in which the calculus of variations is directly relevant. Besides, it is applied in many numerical approaches, grid generation, modern physics, diverse optimization establishments, and flexible dynamics as such. There is a strong relation between calculus of variations and the applications for which variational methods constitute the elemental base. In this way, Nesbet (2003) has emphasized on understanding physical and computational applications of variational methodology rather than on exact mathematical formalism. For him, variational methodology includes electron-impact rotational and vibrational excitation of molecules.

The geodesic equation is a fundamental concept in the geometry of curved spaces, particularly in the context of general relativity. It describes the paths that objects with no force acting on them would naturally follow in curved spacetime. These paths are analogous to straight lines in flat Euclidean space.

The geodesic equation can be derived using the principle of stationary action, also known as the variational principle. The action in this context is a mathematical quantity that describes the dynamics of a system. For the geodesic equation, the relevant action is the so-called "proper time" action, which is related to the time experienced by an observer moving along a certain path.

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Received on Sept. 2, 2023 *Accepted on Dec. 19, 2023* *Published on Jan. 31, 2024*

Here's a brief outline of the derivation using the variational principle:

Define the Action: The action, denoted by S, is defined as the integral of the proper time along the world line of a particle. The proper time τ is the time measured by a clock moving with the particle, and it's related to the spacetime interval ds along the path.

S=∫τ1τ2dτ

Parameterize the Path: Parameterize the path of the particle using some parameter, say λ. The position of the particle in spacetime is described by xμ(λ), where μ is the spacetime index (0 for time, 1, 2, and 3 for spatial coordinates).

Define the Lagrangian: The Lagrangian L is defined as the integrand of the action, i.e.,

$$L = \frac{d\tau}{ds.}$$

This Lagrangian is known as the proper time Lagrangian.

$$L = \frac{1}{2} g\mu\nu \frac{d\lambda^{\mu}}{dx} \frac{dx^{\nu}}{d\lambda}$$

Here, gμν represents the components of the metric tensor, which encodes the geometry of the spacetime.

Variation of the Action: Apply the principle of stationary action, which states that the true path followed by the particle is the one for which the action is stationary (has a stationary value). Mathematically, this is expressed as δS=0 where δS is the variation of the action.

Euler-Lagrange Equation: The variation of the action leads to the Euler-Lagrange equation for each coordinate x^{μ}:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial x^{\mu}}\right) - \left(\frac{\partial L}{\partial x^{\mu}}\right) = 0$$

Here $x^{\mu} = \frac{dx^{\mu}}{d\lambda}$

Geodesic Equation: Simplify the Euler-Lagrange equation for the proper time Lagrangian, and you obtain the geodesic equation:

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

The, $\Gamma^{\mu}_{\alpha\beta}$ terms represent the Christoffel symbols, which are related to the connection or curvature of the spacetime.

In summary, the geodesic equation describes the paths that extremize the proper time action in curved spacetime, and it is a key element in understanding the motion of particles in the presence of gravity as described by general relativity.

Discussion

Variational principle:-

It states: The necessary condition that the integral

$I = \int_{t_1}^{t_2} f(t, x^k, \dot{x}^k)\, dt$ ………… (1)

To be an extremum means (maximum or minimum) point where

F(t,$x^k, \dot{x}^k$) shoud satisfy Euler lagrange equation

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^k}\right) - \frac{\partial F}{\partial x^k} = 0$.................(2)

Def^n:- Geodesic :-

Geodesic is a curve in a given space which makes the minimum distance in that space, for arbitrary displacement the length of the curve remains unchanged provided the end point is kept fixed. E.g.

In surface of sphere Geodesics are of great circle

Derivation:

The distance between two points t_1 and t_2 on a curve$x^k = x^k$(t) in $\int_{t1=(Initialpoint)}^{t2=(finalpoint)} x^k = x^k(t)$

IN Riemann space is given by S=$\int_{t_1}^{t_2}\sqrt{g_{\propto\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}dt}$ ....(3)

Where $g_{\alpha\beta}$ is the fundamental metric tensor?

To find geodesic equation in Riemann space we must determine the minimum of integral (3) by using equation (2) comparing (1) and (3)

We get.

F= $\sqrt{g_{\alpha\beta}\frac{dx^\propto}{dt}\frac{dx\beta}{dt}} = \sqrt{g_{\alpha\beta}\dot{x^\propto}.\dot{x}^\beta}$............ (4)

Now we find.

$\frac{\partial F}{\partial x^k} = \frac{1}{2\sqrt{g_{\alpha\beta}\dot{x}^\propto\dot{x}^\beta}}\left(\frac{\partial g_{\alpha\beta}}{\partial x^k}\right)\dot{x}^\alpha\dot{x}^\beta$..................... (5)

$\frac{\partial F}{\partial \dot{x}^k} = \frac{1}{2\sqrt{g_{\alpha\beta}\dot{x}^\propto\dot{x}^\beta}} 2\, g_{\beta k}\dot{x}^\beta$....................... (6)

Using $\frac{ds}{dt} = \dot{s} = \sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}$

Putting the volume of eq^n (5) and (6) into eq^n (2) we get

$\frac{d}{dt}(\frac{1}{\dot{s}} g_{\alpha\beta}\dot{x}^\beta) - \frac{1}{2\dot{s}}\frac{(\partial g_{\alpha\beta})}{\partial x^k}\dot{x}^\propto\dot{x}^\beta = 0$

$g_{\beta k}\ddot{x}^\beta + \frac{\partial g_{\beta k}}{\partial x^\propto}\dot{x}^\propto\dot{x}^\beta - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^k}\dot{x}^\propto\dot{x}^\beta = \frac{\ddot{s}}{\dot{s}} g_{\beta k}\dot{x}^\beta$..................(7)

Writing $\frac{\partial g_{\beta k}}{\partial x^\propto}\dot{x}^\propto\dot{x}^\beta = \frac{1}{2}\left[\frac{\partial g_{\beta k}}{\partial x^\propto} + \frac{\partial g_{\alpha k}}{\partial x^\beta}\right]\dot{x}^\propto\dot{x}^\beta$

Equation (7) gives.

$$g_{\beta k}\ddot{x}^{\beta}+\frac{1}{2}\left[\frac{\partial g_{\beta k}}{\partial x^{\propto}}+\frac{\partial g_{\alpha k}}{\partial x^{\beta}}-\frac{\partial g_{\alpha\beta}}{\partial x^{k}}\right]\dot{x}^{\propto}\dot{x}^{\beta}=\frac{\ddot{s}}{\dot{s}}g_{\beta k}\dot{x}^{\beta} \;\ldots\ldots\ldots\ldots (8)$$

If we use the length s as a parameter,

$\dot{s}=1$ $\ddot{s}=0$ and Equation (8) becomes .

$$g_{\beta k}\frac{d^2x^{\beta}}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{\beta k}}{\partial x^{\propto}}+\frac{\partial g_{\alpha k}}{\partial x^{\beta}}-\frac{\partial g_{\alpha\beta}}{\partial x^{k}}\right]\frac{dx^{\propto}}{ds}\frac{dx^{\beta}}{ds}=0\ldots\ldots\ldots\ldots\ldots (9)$$

Multiplying both sides by $g^{\rho k}$, we obtain.

$$\delta^{\rho}_{\beta}\frac{d^2x^{\beta}}{ds^2}+\frac{1}{2}g^{\rho k}\left[\frac{\partial g_{\beta k}}{\partial x^{\propto}}+\frac{\partial g_{\alpha k}}{\partial x^{\beta}}-\frac{\partial g_{\alpha\beta}}{\partial x^{k}}\right]\frac{dx^{\propto}}{ds}\frac{dx^{\beta}}{ds}=0$$

Where $g^{\rho k}g_{\beta k}=\ \delta^{\rho}_{\beta}=1\ for\rho=\beta \quad = \quad 0 for \rho\neq\beta$

is called the kronecher delta.

$$\frac{d^2x^{\delta}}{ds^2}+\Gamma^{\rho}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Where $\Gamma^{\rho}_{\alpha\beta}=\frac{1}{2}g^{\rho k}\left[\frac{\partial g_{\alpha k}}{\partial x^{\beta}}+\frac{\partial g_{\beta k}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha\beta}}{\partial x^{k}}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$

Is called affine connection and eq^n (10) is the required equatipn of geodesic in Riemannian space.

Application:

Shortest distance between any two points in Euclidean space is the straight line.

In Euclidean space affine connection defined in

Equation (11) vanish. ($\therefore g_{\alpha\beta}$ = constant)

So that equation (10) reduces to

$$\frac{d^2x^{\rho}}{ds^2}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Let us consider a plane for which x^{ρ} = y (displacement)

And the paramenter s = t (time) then eq^n (12) becomes

$$\frac{d^2y}{dt^2}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Integrating equation (13) two times we get

Y = mt + c ……………………….(14)

Where m and c are the constatant of integration, m is also called the slope and eq^n (14) is the straight line.

The geodesic equation can be derived from a variational principle known as the principle of stationary action. In the context of general relativity, this principle is applied to the action functional for a free-falling particle. The action is defined as the integral of the proper time along the world line of the particle. The proper time is an invariant quantity that measures the elapsed time experienced by an observer moving along the world line.

The action functional S is given by:

$S= -m\int d\tau$

Where m is the mass of the particle, τ is the proper time, and the integral is taken along the world line.

The geodesic equation can be obtained by extremizing this action with respect to the path of the particle. Mathematically, this is expressed as the variation of the action being set to zero:

$\delta S=0$

To perform the variation, one considers small perturbations in the trajectory of the particle. The action variation is then expressed in terms of the perturbations in the trajectory and its derivatives with respect to the proper time.

After performing the variation and simplifying the expression, the resulting equation is the geodesic

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

Here, x^μ are the spacetime coordinates $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols representing the connection coefficients, and τ is the proper time.

This second-order differential equation describes the motion of a free-falling particle in a curved spacetime, and it is the geodesic equation that embodies the principle of stationary action for the proper time integral.

Conclusion

The paper has found out that there is a connection between variational principle and variational calculus. The strategy of the variational principle uses a problem to solve an approximate problem that is unsolvable. From the variational principle of action, many equations of mathematical physics can be derived. Thus, variational calculus supplies the analytic connection bridging ancient conjectures concerning an ideal universe with contemporary demands for optimal control of operating systems. It is influential in formulating variational principles of mechanics and physics and resumes insight into the relationship between these principles. Overall, geodesics on a bowed surface connecting two points can be structured by expanding a thread between the very points.

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