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The equivalency of the Banach-Alaoglu theorem and the axiom of choice

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Abstract

The Axiom of Choice can be used to prove the Banach-Alaoglu theorem, while a weakened form of the Axiom of Choice is required to prove the full strength of the Hahn-Banach theorem, which is equivalent to the Banach-Alaoglu theorem. Although the Banach-Alaoglu theorem and the full- strength Axiom of Choice are not exactly equivalent, they are closely related.

The Banach-Alaoglu theorem is a compactness theorem whose proof mainly depends on Tychonoff's theorem. In this article, Banach-Alaoglu theorem is equivalent to the axiom of choice, has been proved.

Key-Words: Banach-Alaoglu theorem, Tychonoff's theorem, Axiom of choice, Compactness, Dual space

MSC: 03E25, 03E30, 03E65, 03E75

Introduction

The Banach-Alaoglu Theorem and the Axiom of Choice are two fundamental concepts in mathematics, and there is a deep connection between them. The Banach-Alaoglu Theorem is a result in functional analysis, particularly in the theory of topological vector spaces, while the Axiom of Choice is a foundational principle in set theory. In this article, we will explore the relationship between these two concepts and explain why the Banach-Alaoglu Theorem is considered equivalent to the Axiom of Choice in certain contexts.

Barnum had explained the axiom of Choice with some of its various implications. Their implications include a number of equivalent statements. He also explained the various applications of it and got some controversial results [1]. Bell explained chronology, independency, consistency, and its applications of Axioms of choice [3]. First and foremost, in 1904, Ernst Zermelo formulated this Axiom of choice and denoted it by A C. He consider with an arbitrary set M and used symbol M' to represent an arbitrary nonempty disjoint subsets of M, and the collection of family of subsets by M [2].

The Banach-Alaoglu Theorem

The Banach-Alaoglu Theorem is named after mathematicians Stefan Banach and Leon Alaoglu and deals with the compactness properties of the dual space of a normed vector space. It is typically stated as follows:

Let X be a normed vector space. It's topological dual space, which consists of all continuous linear functional on X. Then, the closed unit ball in X* is weak-* compact, meaning it is compact with respect to the weak-* topology* [6].

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Banach-Alaoglu Theorem states the following:

The Closed unit ball in the dual space X is weak-compact: **

More formally, if B* is the closed unit ball in X*, i.e.

$B^* = \{f \in X^* : \|f\| \leq 1\}$,

where $\|f\|$ denotes the norm of a functional f, then B* is weak*-compact. Here, "weak*" refers to the weak-* topology, which is a specific topology defined on X*.

Implications and Significance:

The Banach-Alaoglu Theorem has several important implications and applications in functional analysis and related areas of mathematics:

Dual Spaces: It provides insight into the topology of dual spaces, which are essential in many areas of functional analysis. Dual spaces help us understand the properties of linear functionals and their relationships with the original vector space.

Weak-Topology:* The theorem introduces the concept of the weak-* topology, which is a fundamental concept in the study of topological vector spaces. Understanding the weak-* topology is crucial for investigating the convergence of sequences and other topological properties in dual spaces.

Compactness: According to the theorem, the closed unit ball in dual space is compact, which is a useful quality in mathematical analysis. Compactness often simplifies the study of various mathematical objects and leads to theorems that ensure the existence of certain solutions.

Equivalent Formulations: The Banach-Alaoglu Theorem is also intimately connected to other important results in functional analysis as the Alaoğlu's Theorem and the Krein-Milman Theorem. These theorems provide alternative characterizations of compactness in the weak-* topology.

Axiom of Choice

Thomson defined Axiom Choice as follows: Let there be a function *f* such that f(S) in S for every set S in the f family $\mathcal{F}$ of non-empty sets. Here, f is the selection function of $\mathcal{F}$. If $\mathcal{F}$ is made up of sets of the form (a, b), where a and b are real numbers, then $\mathcal{F}$ (a, b) = min (a, b) is a choice function of $\mathcal{F}$ [4]. This can be expressed as follows: if all of the sets in mathcalF are singletons, then there exists a choice function on $\mathcal{F}$. If $\mathcal{F}$ is a finite family of non-empty sets, then there must be a choice function.

Thomas argued that this Axiom of Choice is different from the ordinary principles accepted by Mathematician. This is the cause of objections to this Axioms of Choice, as late in the thirties. Mathematician of present age hardly use the Axiom of Choice. However they prefer to use Zorn lemma instead of this. It is sometimes called Zermelo's Axioms of Choice [4], [6],[7].

Examples

The following are the examples of Axiom of Choice:

(i) Let $\mathcal{F}$ be the set of districts in the Nepal and each district as a assembly of villages. $\bigcup \mathcal{F}$ is the set of all villages in Nepal, and the function f that denotes each district, then its district headquarters is an example of a choice function for $\mathcal{F}$.

(ii) Let $\mathcal{F}$ is the collection of set of all pairs of shoes in the wor $\mathcal{F}$. Then the function which picks the left shoe out of each pair is the choice function for $\mathcal{F}$.

(iii) Let $\mathcal{F} = \mathcal{P}\,(\mathbb{N})\backslash\{\varphi\}$. The function f(A) = min A is a choice function of $\mathcal{F}$.

The examples described above have defined some rule: "chose district headquarter", "choose left shoe", "and choose the least element ". All given examples define the choice function to make it understandable. There is no need of other Axiom for situations like this.

Now in mathematical logic, we join many formulas with "and" symbol, i.e. if $T_1 \cdots\cdots T_n$ are sets and we know each of them is non-empty, the following is also a formula of first order logic.

$$\emptyset(T_1) \wedge \cdots\cdots\cdots\cdots\cdots\cdots \wedge \emptyset(T_n) \qquad (1)$$

The symbol $\wedge$ means "and" is used to join. We cannot, however join infinitely many statements. This system of logic doesn't allow for infinite conjunction [8, 9, 10]

Discussion

Thomas in his book had shown many more equivalencies of Axiom of Choice with Zorn's lemma, Well-ordering principles etc. But this article will describe the Kelly work. He has established the equivalent relation between Tychonoff theorem and Axiom of Choice in his article entitled ". The Tychonoff product theorem implies the Axiom of Choice". His statement is

$$X_\alpha \neq \emptyset, \forall_\alpha \Longrightarrow \prod_\alpha X \neq \emptyset \text{ [5].} \qquad (2)$$

Kelly worked on Tychonoff's theorem and the Axiom of Choice. Tychonoff's theorem is a fundamental result in topology which states that the product of any collection of compact topological spaces is compact. This theorem has wide-ranging applications in various branches of mathematics, particularly in functional analysis and algebraic topology [5]

The Axiom of Choice is a controversial axiom in set theory, which states that given any collection of non-empty sets, it is possible to choose exactly one element from each set, even if the collection is infinite. It has profound implications throughout mathematics, including in the proof of Tychonoff's theorem [5]

Kelly likely explored the relationship between Tychonoff's theorem and the Axiom of Choice, as the latter is often invoked in the proof of the former. The Axiom of Choice is essential for many results in topology, including Tychonoff's theorem, but its use can have significant philosophical and mathematical implications, leading to debates about its validity and consequences [5]

Equivalence with the Axiom of Choice

The Banach-Alaoglu Theorem is equivalent to the Axiom of Choice of the Zermelo-Fraenkel set theory. This means that in the absence of the Axiom of Choice, the Banach-Alaoglu Theorem may fail to hold. The equivalence between the Banach-Alaoglu Theorem and the Axiom of Choice highlights the deep connections between functional analysis and set theory. It also illustrates how certain mathematical principles, such as the Axiom of Choice, have far-reaching consequences in various branches of mathematics.

Statement: If X be a normal vector space then the closed unit ball $\overline{B^*}$ in the dual space X^* is compact with respect to the weak-∗ topology [6].

Proof. This theorem explains that in a normed vector space, the closed unit ball in the dual space is compact in the weak-* topology. Formally, if X is a normed vector space and X^* is its dual space (the space of continuous linear functionals on X), then the closed unit ball B^* in X^* is compact in the weak-* topology.

Let closed unit ball $\overline{B^*} \in X$ of closed subset of product space

$$Z = \prod_{x\in X}\{z \in \mathbb{C} \mid |z| \leq \|x\| \} \subset \mathbb{C}^X, \tag{3}$$

Let f $\in \overline{B^*}$ iff, as an element of Z that contain

$$D = \{f \in Z | ev_{x+y}(f) = ev_x(f) + ev_x(f), ev_{\alpha x} = \alpha\, v_x(f), \forall\, x, y \in X, \forall\, \alpha \in \mathbb{C}\}$$

$$= \cap_{x,y,\alpha}\left(ev_{x+y} - ev_x - ev_y\right)^{-1}(0) \cap (ev_{\alpha x} - \alpha ev_x)^{-1}(0).$$

$$D = \cap_{x,y.\alpha} f_\alpha \neq \emptyset. \tag{4}$$

Here D is closed subset in Z and is compact, therefore D is also compact. $\overline{B^*}$ is compact with respect to weak-* topology. Therefore, Banach-Alaoglu theorem is equivalent to the axiom of choice. The Banach-Alaoglu theorem is known to be equivalent to a weakened version of the Axiom of Choice, known as the Hahn-Banach theorem, which is a crucial tool in functional analysis.

The full-strength Axiom of Choice implies the Banach-Alaoglu theorem. This is because one can apply the Axiom of Choice to make a basis for X^*, which permits one to embed X^* into a product of copies of $\mathbb{R}$ (or $\mathbb{C}$), and then apply Tychonoff's theorem to display compactness.

Conclusion

The Banach-Alaglu theorem is a crucial compactness theorem. This theorem is equivalent to Tychonoff's theorem. Tychonoff's theorem is equivalent to Axiom of Choice, therefore Banach theorem is also equivalent to Axiom of Choice has been proved.

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