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q-HERMITE-HADAMARD INTEGRAL INEQUALITY FOR THE COORDINATED CONVEX FUNCTIONS

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Abstract.

The calculus without the notion of limits is quantum calculus. Its study dates back to L. Euler in the middle of the eighteenth century whereas the systematic initiation on it was done by F.H. Jackson in the beginning of the twentieth century. The rapid growth on q-calculus is due to its applications in various branches of mathematical and physical streams. Of them, one of the most basic and important functions in the theory of geometric function is convexity having its wider applications in pure and applied mathematics. As it still lacks the intensive study on quantum estimates on the various types of integral inequalities, we focus our study on quantum estimates of Hermite-Hadamard type integral inequality especially on coordinated convex functions. In this paper, we have extended Hermite-Hadamard type integral inequality for coordinated convex function in terms of quantum framework.

Key Words: Convexity, coordinated convexity, Hermite-Hadamard inequality, q-Calculus.
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1. INTRODUCTION

The modern name for the investigation of calculus without limits is quantum calculus. The pioneering work on q- calculus was done by L. Euler on Newton's work on infinite series in 1748. Later on, it was enhanced by Gauss and Heine by introducing q-hyper-geometric function. Multiple areas of mathematics and physics, including number theory, fundamental hyper-geometric functions, combinatorics, orthogonal polynomials, mathematical inequalities, quantum theory, mechanics, and the theory of relativity, all have numerous uses for the subject of quantum calculus, and, hence the q-calculus is appeared as an inter-disciplinary subject between mathematics and physics, a blend of the subjects. For detail see these [3, 4] The systematic study on q-calculus is made by F.H. Jackson in the beginning of the 20th century by introducing q-derivative and q-integral. In 1908, F.H. Jackson reintroduced the Euler- Jackson q-difference operator as follows:

Definition 1.1. [4] The q-derivative of a continuous function f is defined as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in (0, b), \quad q \in \mathbb{C}\backslash\{1\} \tag{1.1}$$

for $f : (0, b) \to \mathbb{R}, \quad 0 \leq b < \infty$.

In 1910, Jackson also introduced the concept of q-definite integral extending the idea of q-definite integral as follows:

Definition 1.2. [4] The q- definite integral is defined as

$$\int_0^x f(t) d_q t = (1-q)x \sum_0^\infty q^n f(q^n x), \quad x \in (0, \infty).$$

Tariboon et. al in 2013 [7] defined the q-derivative in a finite interval as follows:

Definition 1.3. The q-derivative of a continuous $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ at a point $t \in J$ on $[a, b]$ is defined as

$${}_aD_q f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-t)(1-q)}, \quad t \neq a$$

and,

$${}_aD_q f(a) = \lim_{t \to a} aD_q f(t).$$

If $a=0$, then ${}_0D_qf = D_qf$, where D_q is well known q-derivative of the function $f(t)$ defined as in (1.1)

Tariboon et. al[7] defined q- integral in a finite interval $J=[a,b]$ as follows.

Definition 1.4. Let $f : J \to \mathbb{R}$ be a continuous function. Then the q -integral on J is defined by

$$\int_a^b f(t)\,{}_ad_qt = (1-q)(a-b)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b)$$

for $x \in J$.

Now, we define the convex function as follows:

Definition 1.5. [6] Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a real valued function. Then, the function f is said to be a convex function, if the inequality

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$$

holds, for all $a,b \in I$, and, $t \in [0,1]$

If $t=\frac{1}{2}$, then the convex function f satisties the following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$$

for all $a,b \in I$ and $t \in [0,1]$ which is called Jensen's inequality.

Next, we define the convexity of the function f in coordinates as follows:

Definition 1.6. A function $f : [a,b]\times[c,d] = \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds:

$$f(tx+(1-t)z, ty+(1-t)w) \leq tf(x,y)+(1-t)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$, $t \in [0,1]$

Dragomir [2] reintroduced the convexity of the function f on coordinates as follows:

Definition 1.7. Let us consider the bi-dimensional interval $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2$ with $a<b$, $c<d$. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the coordinates if the partial mapppings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex for all $y \in [c,d]$, $x \in [a,b]$.

Latif et. al [5] gave a more formal definition of coordinated convex function which is stated as follows:

Definition 1.8. A function $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$ is a coordinated convex on Δ if the following inequality holds for all $s,t \in [0,1]$

$$f(tx + (1-t)y, sz + (1-s)w) \leq tsf(x,z) + t(1-s)f(x,w) + s(1-t)f(y,z) + (1-t)(1-s)f(y,w)$$

2. PRELIMINARY RESULTS

One of the most famous and intensively studied inequalities in literature for the class of convex function, a necessary and sufficient condition for the function to be convex, is the Hermite-Hadamard inequality which is stated as follows:

Theorem 2.1. *[8] Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval $[a,b]$ of real numbers $a,b \in \mathbb{R}$ with $a < b$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

The inequalities hold in reverse direction if the function f is concave.

Tariboon et al [8] in 2014 extended HH inequality in q framework as follows: Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and $0 < q < 1$ be a constant. Then, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

Kunt and Iscan in 2016 gave a counter example to the above inequality and showed that the left-hand side of the above inequality is not correct, and provided the correct form of q-Hermite-Hadamard inequality which is stated as follows:

Theorem 2.2. *[1] Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$ be a constant. Then, we have*

$$f\left(\frac{qa+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

Alp et al in 2018 presented the generalized q-Hermite- Hadamard integral inequality as follows:

Theorem 2.3. *[1] Let $f : [a,b] \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$ be a constant. Then, we have*

$$max\{I_1, I_2, I_3\} \leq \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

where,

$$\begin{aligned} I_1 &= f\left(\frac{qa+b}{1+q}\right) \\ I_2 &= f\left(\frac{a+qb}{1+q}\right) + \frac{(1-q)(b-a)}{1+q} f'\left(\frac{a+qb}{1+q}\right) \\ I_3 &= f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right) \end{aligned}$$

S.S. Dragomir [2] in 2001 presented the Hermite-Hadamard inequality for the coordinated convex functions follows:

Theorem 2.4. *Let $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a coordinated convex function on Δ. Then,*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b (f(x,c)+f(x,d))\,dx + \frac{1}{d-c}\int_c^d (f(a,y)+f(b,y))\,dy\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

In this paper, we extend Hermite-Hadamard type integral inequality for the coordinated convex function which was established by Dragomir in terms of in q- calculus.

3. MAIN RESULTS

The q-Hermite-Hadamard type integral inequality for coordinated convex function is established in the following theorem.

Theorem 3.1. *Let $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a coordinated convex function on*

Δ. *Then,*

$$f\left(\frac{qa+b}{1+q},\frac{qc+d}{1+q}\right)\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{qc+d}{1+q}\right)\,_a d_q x+\frac{1}{d-c}\int_c^d f\left(\frac{qa+b}{1+q},y\right)\,_c d_q y\right]$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,_a d_q x\,_c d_q y$$
$$\leq\frac{q}{2(1+q)(b-a)}\int_a^b f(x,c)\,_a d_q x+\frac{1}{2(1+q)(b-a)}\int_a^b f(x,d)\,_a d_q x$$
$$+\frac{q}{2(1+q)(d-c)}\int_c^d f(a,y)\,_c d_q y+\frac{1}{2(1+q)(d-c)}\int_c^d f(b,y)\,_c d_q y$$
$$\leq\frac{q^2 f(a,c)+qf(a,d)+qf(b,c)+f(b,d)}{(1+q)^2}.$$

The above inequalities are sharp.

Proof. As $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is convex on the coordinates, it follows that the mapping

$$g_x:[c,d]\to\mathbb{R},\quad g_x(y)=f(x,y)$$

is convex on $[c,d]$ for all $x\in[a,b]$ and $0<q<1$ be a constant. By using q-Hermite-Hadamard integral inequality, we have

$$g_x\left(\frac{cq+d}{1+q}\right)\leq\frac{1}{d-c}\int_c^d g_x(y)_c d_q y\leq\frac{qg_x(c)+g_x(d)}{1+q} \tag{3.1}$$

that is

$$f\left(x,\frac{cq+d}{1+q}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)_c d_q y\leq\frac{qf(x,c)+f(x,d)}{1+q} \tag{3.2}$$

By q integrating with respect to x on $[a,b]$ to the inequality (3.2), we have

$$\frac{1}{b-a}\int_a^b f\left(x,\frac{cq+d}{1+q}\right)\,_a d_q x\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_c d_q y_a d_q x\leq\int_a^b\frac{qf(x,c)+f(x,d)}{1+q}\,_a d_q x \tag{3.3}$$

Similarly, the mapping $g_y:[a,b]\to\mathbb{R},\quad g_y(x)=f(x,y)$ is convex on $[a,b]$ for all $y\in[c,d]$

and $0 < q < 1$ be a constant. By using q-Hermite-Hadamard's inequality, we have

$$g_y\left(\frac{aq+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b g_y(x)_a d_q y \leq \frac{qg_y(a)+g_y(b)}{1+q} \tag{3.4}$$

that is

$$f\left(\frac{qa+b}{1+q}, y\right) \leq \frac{1}{b-a}\int_a^b f(x,y)_a d_q y \leq \frac{qf(a,y)+f(b,y)}{1+q} \tag{3.5}$$

By q integrating with respect to y to the above inequality on $[c,d]$, we have

$$\frac{1}{d-c}\int_c^d f\left(\frac{qa+b}{1+q}, y\right) {}_c d_q y \leq \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)_c d_q y_a d_q x \leq \int_c^d \left(\frac{qf(a,y)+f(b,y)}{1+q}\right) {}_c d_q y \tag{3.6}$$

Now, summing up the inequalities (3.3) and (3.6), we have

$$\begin{aligned}
\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{cq+d}{1+q}\right) + \frac{1}{d-c}\int_c^d f\left(\frac{qa+b}{1+q}, y\right)\right] \\
\leq \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)_c d_q y_a d_q x \\
\leq \frac{1}{2}\left[\int_a^b \frac{qf(x,c)+f(x,d)}{1+q} {}_a d_q x + \int_c^d \left(\frac{qf(a,y)+f(b,y)}{1+q}\right) {}_c d_q y\right] \\
\leq \frac{q}{2(1+q)(b-a)}\int_a^b f(x,c)\,{}_a d_q x + \frac{1}{2(1+q)(b-a)}\int_a^b f(x,d)\,{}_a d_q x + \frac{q}{2(1+q)(d-c)}\int_c^d f(a,y)\,{}_c d_q y \\
+ \frac{1}{2(1+q)(d-c)}\int_c^d f(b,y)\,{}_c d_q y
\end{aligned} \tag{3.7}$$

Finally, using q Hermite-Hadamard inequality, we have

$$\frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \leq \frac{qf(a)+f(b)}{1+q} \tag{3.8}$$

that is

$$\frac{1}{b-a}\int_a^b f(x,c)\,{}_a d_q x \leq \frac{qf(a,c)+f(b,c)}{1+q} \tag{3.9}$$

Thus, we have

$$\frac{q}{2(1+q)(b-a)}\int_a^b f(x,c)\,{}_ad_qx = \frac{q}{2(1+q)}.\frac{qf(a,c)+f(b,c)}{1+q} = \frac{q^2f(a,c)+qf(b,c)}{2(1+q)^2} \quad (3.10)$$

Also, we have

$$\frac{1}{2(1+q)(b-a)}\int_a^b f(x,d)\,{}_ad_qx = \frac{1}{2(1+q)}\frac{qf(a,d)+f(b,d)}{1+q} = \frac{af(a,d)+f(b,d)}{2(1+q)^2} \quad (3.11)$$

Similarly, we obtain

$$\frac{q}{2(1+q)(d-c)}\int_c^d f(a,y)\,{}_cd_qy = \frac{q^2f(a,c)+qf(a,d)}{2(1+q)^2} \quad (3.12)$$

And, we also have in similar fashion that

$$\frac{1}{2(1+q)(d-c)}\int_c^d f(b,y)\,{}_cd_qy = \frac{qf(b,c)+f(b,d)}{2(1+q)^2} \quad (3.13)$$

On summing up $(3.10),(3.11),(3.12)$ and (3.13), we have

$$\begin{aligned}\frac{q}{2(1+q)(b-a)}\int_a^b f(x,c)\,{}_ad_qx + \frac{1}{2(1+q)(b-a)}\int_a^b f(x,d)\,{}_ad_qx \\ + \frac{q}{2(1+q)(d-c)}\int_c^d f(a,y)\,{}_cd_qy + \\ \frac{1}{2(1+q)(d-c)}\int_c^d f(b,y)\,{}_cd_qy \\ \leq \frac{q^2f(a,c)+qf(b,c)+qf(a,d)+f(b,d)}{(1+q)^2}\end{aligned}$$

This completes the proof. □

Remark 3.2. If $q \to 1$, then the above inequalities reduces to Dragomir's result on Hermite-Hadamard integral inequality as stated in theorem (2.4).

4. CONCLUSION

In this paper, we have been able to extend Hermite-Hadamard type integral inequality for coordinated convex function in terms of q framework. This result definitely helps to determine the bounds of the integral mean of the coordinated convex function in terms of

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