अवधारणा AWADHARANA
[A Peer-Reviewed, Open Access Journal]
ISSN : 2505-080x (Print)
Volume 8, September 2024
Published by Creative Democrat Academy

Tensors, their variants, and operations

Iswar Mani Adhikari, PhD
Associate Professor, Department of Mathematics
Prithvi Narayan Campus, Pokhara

Article History : Submitted 19 April 2024; Reviewed 3 June 2024; Accepted 11 July. 2024
Author : Iswar Mani Adhikari Email: adhikariim35@gmail.com

Abstract

Tensors are mathematical objects that can describe physical properties like scalars and vectors. Tensors are the generalization of scalars and vectors; a scalar is a zero-rank tensor, and a vector is a first-rank tensor where the rank or the order of a tensor is defined by the number of directions required to describe it. In this paper, the focus is to insight into different fundamental tensors, their variants, and operations.

Keywords: Tensors, contravariant tensor, covariant tensor, rank of tensors, operations with tensors.

1. Introduction

The word tensor was introduced in 1846 by William Rowan Hamilton and later used in different mathematical branches like differential geometry, algebra, calculus, mechanics, topology, etc. Tensors and tensor fields were found useful in different fields like Einstein's theory of general relativity, homological algebra, category theory, quantum physics, engineering, machine learning, neural networks, etc. Gregorio Ricci along with his student Tullio Levi Civita was considered the first one to develop a tensor analysis for Physics. This is why the tensor calculus is also known as Ricci Calculus. Tensors are developed to formally study and manipulate the geometrical patterns during the study and analysis of mathematical curves and surfaces as a vector extension. The stress inside a solid is expressed as a tensor field. The stress and strain tensors are the second order tensors whereas the elasticity is under the fourth order tensor field. For further information see (Eslami 2024, Hamilton 2001; Hess 2019; Islam, 2006; Ji *et al.* 2019).

An ordered set of n real variables can be denoted by $(x^1, x^2, x^3, \dots x^n)$ where 1,2,3, …, n is not taken as the exponents but are the superscripts as the frame of references. In three-dimensional space, a point is a set of three numbers as a coordinate and that can be rectangular, cylindrical, or spherical coordinates, etc.

Let $(x^1, x^2, \dots, x^n)$ and $(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^n)$ be coordinates of a point in two different frames of reference. Then there exist n different independent relations between the coordinates of the two systems with the forms like:

$$\overline{x}^1 = \overline{x}^1(x^1, x^2, \dots, x^n)$$

$$\overline{x}^2 = \overline{x}^2(x^1, x^2, \dots, x^n)$$

$$\dots \quad \dots \quad \dots$$

$$\overline{x}^n = \overline{x}^n(x^1, x^2, \dots, x^n)$$

It can be indicated simply as,

$$\overline{x}^k = \overline{x}^k(x^1, x^2, \dots, x^n) \quad \text{for } k=1, 2, \dots, n. \qquad (1)$$

For such a set of coordinates having the functions to be single-valued, continuous, and continuous derivatives, there will correspond a unique set $(x^1, x^2, \dots, x^n)$given by,

$$x^k = x^k(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^n) \text{for } k=1, 2, \dots, n. \qquad (2)$$

These relations (1) and (2) define a transformation of coordinates from one frame of reference to another.

We have some Einstein's summation convention**,** according to Goodstein (1982):

(i) An index, subscript, or superscript that appears exactly twice in any term of an expression is to be summed over all the values that an index in that position can take.

(ii) Such indexed quantities may occur in the numerator or/and denominator of a form of the expression.

(iii) The ranges of the values for such subscripts or superscripts must be specified. If not, they are apparent from the context, in general.

(iv) Those subscripts or superscripts that are summed over are named dummy subscripts and the rest are free subscripts.

Example 1: Consider the three-dimensional space R^3. Then,

(i) $\sum_{i=1}^{n} a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3$. i.e., $a_i x^i$ or $a_m x^m$, dummy index.

(ii) $a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k}$

(iii) $a_{ij} b_{jk} = \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ij} b_{jk} c_k$

(iv) $\frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$

(v) $\frac{\partial^2 \phi}{\partial x_i \partial x_i} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$

(vi) $a_{ij} b_{jk} c_{kl} \neq a_{ij} b_{jj} c_{jl}$ (k replaced by j, i.e. by such script which is already present). And hence, $a_{ij} b_{jk} c_{kl} \neq a_{il} b_{lk} c_{kl}$.

(vii) However, we can write, $a_{ij} b_{jk} c_{kl} = a_{im} b_{mk} c_{kl}$ (k replaced by j),

Equivalent to say $a_{ij} b_{jk} c_{kl} = a_{im} b_{mk} c_{kl}$ (k replaced by j).

An index occurring only once in each term is named the free index and can stand for any of the numbers 1, 2, …, n such as k in Equation (1) and (2).

2. Fundamental tensors and their variants

Here, we present the fundamental tensors of different ranks and their basic variants. For further information see (Comon, 2014; Guo 2021).

2.1 Contravariant and covariant tensors of the first order

If n quantities $P^1, P^2, \ldots, P^n$ in a frame of reference $(x^1, x^2, \ldots, x^n)$ are related to n other quantities $(\overline{P}^1, \overline{P}^2, \ldots, \overline{P}^n)$ in another coordinate system $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$ by the transformation equations like,

$$\bar{P}^{p} = \sum_{\kappa=1}^{n} \frac{\partial \bar{x}^{p}}{\partial \bar{x}^{\kappa}} P^{\kappa} \text{ for p= 1, 2, …, n.}$$

It can be expressed as, $\bar{P}^{p} = \frac{\partial \bar{x}^{p}}{\partial \bar{x}^{\kappa}} P^{\kappa}$ (3)

This equation (3) gives the components of a contravariant tensor of the first rank or the first order. A similar result can be obtained for the components of the covariant tensor.

For this, if n quantities $P_1, P_2, \dots, P_n$ in a frame of reference $(x^1, x^2, \dots, x^n)$ are related to n other quantities $(\overline{P_1}, \overline{P_2}, \overline{P_3}, \dots, \overline{P_n})$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ by the transformation equations like,

$$\bar{P}_p = \sum_{\kappa=1}^{n} \frac{\partial \bar{x}^{\kappa}}{\partial \bar{x}^{p}} P_k \text{ for p= 1, 2, …, n.}$$

It can be expressed as $\bar{P}_p = \frac{\partial \bar{x}^{\kappa}}{\partial \bar{x}^{p}} P_k$ (4)

This equation (4) gives the components of a covariant tensor of the first rank or the first order. Note that a superscript and subscript are to indicate the contravariant and the covariant components, respectively.

2.2 Contravariant, covariant, and mixed tensors of second order

Let n^2 quantities $P^{\kappa r}$ in a coordinate system $(x^1, x^2, \dots, x^n)$ are related to n^2 quantities $\bar{P}^{ps}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ by the transformation equations like,

$$\bar{P}^{ps} = \sum_{r=1}^{n} \sum_{k=1}^{n} \frac{\partial \bar{x}^{p}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{s}}{\partial \bar{x}^{r}} P^{kr} \text{ for p, s= 1,2, …, n .}$$

For simplicity, such contravariant components of a tensor of rank two can be expressed as,

$$\bar{P}^{ps} = \frac{\partial \bar{x}^{p}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{s}}{\partial \bar{x}^{r}} P^{kr} \quad (5)$$

The n^2 quantities P_{kr} are the covariant components of a tensor of the second rank can be expressed as,

$$\bar{P}_{ps} = \frac{\partial x^{k}}{\partial \bar{x}^{p}} \frac{\partial x^{r}}{\partial \bar{x}^{s}} P_{kr} \quad (6)$$

Similarly, the n^2 quantities P_s^k are called components of a mixed tensor of second rank or second order and can be expressed as

$$\bar{P}_s^p = \frac{\partial \bar{x}^p}{\partial x^k} \frac{\partial x^r}{\partial \bar{x}^s} P_s^k \quad (7)$$

2.3 Mixed tensor of rank greater than two

Likewise, one can define the tensors of rank greater than two. For example, P_{nl}^{krt} are the components of a mixed tensor of rank 5, contravariant of order 3, a

$$\bar{P}_{ij}^{psm} = \frac{\partial \bar{x}^p}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^r} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} P_{nl}^{krt} \quad (8)$$

2.4 Kronecker delta

In such metric tensors, the Kronecker tensor, simply a Kronecker delta has a wide application, and such a symbol can be defined as,

$\delta_i^j = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$ Note that $\delta_i^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_n^n = 1 + 1 + \cdots + 1 = n$

And $\frac{\partial x^i}{\partial x^j} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$

It is a simple mixed tensor of second rank. Similarly, it can also be defined as,

$\delta^{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$ and $\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$

Such Kronecker delta can be used to replace one subscript with another subscript in certain expressions. We have a simple illustration:

Example 2 The dummy index shared by both terms can be replaced by a free index of such Kronecker delta and finally the delta symbol can be disappeared. Mathematically,

$a_{im}\delta_{mk} = a_{im} \; \delta_{km} = a_{ik}.$

2.5 Symmetric and skew-symmetric tensor

Consider a tensor with any two contravariant or covariant indices. It becomes symmetric if the components remain unaltered upon the interchange of its indices. For example, $P_{kr}^{mps} = P_{kr}^{pms}$, it is symmetric in m and p. Moreover, if a tensor is symmetric concerning any two contravariant and any two covariant indices, it is called symmetric. Likewise, for skew-symmetric we have, $P_{kr}^{mps} = -P_{kr}^{pms}$.

3. Fundamental operations on tensors.

Here, we present some fundamental operations on tensors like addition; subtraction; multiplication as the outer and inner; contraction; quotient law; metric, reciprocal, and associative tensors in brief. We refer to (Comon, 2014; Guo 2121; Ji *et al*. 2019) for details.

3.1 Addition

The sum of two or more tensors of the same rank and type is also a tensor of the same rank and type. Such an addition is commutative and associative too. Let P_k^{mp} and Q_k^{mp} are two tensors then $A_k^{mp} = P_k^{mp} + Q_k^{mp}$ is also a tensor.

3.2 Subtraction

The difference of two tensors of the same rank and type is also a tensor of the same rank and type. Let P_k^{mp} and Q_k^{mp} are tensors then $S_k^{mp} = P_k^{mp} - Q_k^{mp}$ is also a tensor.

3.3 Outer multiplication

The product of two tensors is a tensor in which the rank is the sum of the ranks of its component tensors. In such a product, the ordinary multiplication of the components is carried out and is named an outer product. Let P_k^{rp} and Q_s^m are two tensors then their outer product is given by $P_k^{rp} Q_s^m = R_{ks}^{rpm}$. Nevertheless, every tensor can be expressed as a product of two tensors of lower rank. However, the division of tensors is not always possible.

3.4 Inner multiplication

The outer product of two tensors P_k^{rp} and Q_{st}^m is given by $P_k^{rp} Q_{st}^m$. Letting k=m, we obtain the inner product be $P_m^{rp} Q_{st}^m$. Again, letting k=m and p=s, another inner product becomes $P_m^{rp} Q_{pt}^m$. The inner product of tensors is commutative and associative.

3.5 Contraction

If one covariant and a contravariant index of a tensor are set equal and the summation over the equal indices is carried out as in summation convention, then the resulting sum becomes a tensor of rank two less than the original tensor. Such a process is called contraction. For example, let a mixed tensor of rank 5 be P_{n}^{krt}. Set t=s to obtain $P_{nt}^{krt} = Q_{n}^{kr}$ a tensor of rank 3. Again, for r=n, we get $Q_{n}^{kr} = R^{k}$ a tensor of rank 1 as a contraction.

3.6 Quotient law

Consider quantity X, it may or may not be a tensor. If its inner product with an arbitrary tensor is itself a tensor, then X is also a tensor. This is called quotient law.

3.7 Metric tensors

Consider a rectangular coordinate system as (x, y, z), then the arc length is given by, $ds^2 = dx^2 + dy^2 + dz^2 = \sum_{r=1}^{3} \sum_{k=1}^{3} g_{rk}\ du_r\ dv_k$. The line element ds in such space is given in the quadratic form, more generally as the fundamental quadratic form or the metric form, and is expressed as,

$$ds^2 = \sum_{r=1}^{n} \sum_{k=1}^{n} g_{rk}\ dx^r\ dx^k.$$

By summation convention, we can write it as $ds^2 = g_{rk}\ dx^r\ dx^k$, where g_{rk} represents the metric tensor. The fundamental metric form extended to higher dimensional space is of special importance in the theory of relativity.

The metric tensor is symmetric, i.e., $g_{rk} = g_{kr}$. If $g_{rk} = 0\ , r \neq k,$ then the coordinate system becomes orthogonal.

3.8 Reciprocal tensors

Let $g = |g_{rk}|$ denote the determinant with the elements g_{rk} and consider $g \neq 0$.

Define $g^{rk} = \frac{cofactor\ \ of\ g_{rk}}{g}$ then such g^{rk} is a symmetric contravariant tensor of rank two and is called the reciprocal or the conjugate tensor of g_{rk}. Moreover, $g^{rk}\, g_{pk} = \delta_{p}^{r}$.

3.9 Associated tensors

Let a tensor be given. Then we can derive other tensors by raising or lowering the indices and are simply the associated tensors. Forming inner products of the given tensor will also give the associated tensors.

4. Conclusions

A scalar quantity requires just a number whereas a vector quantity requires generally three numbers for its specification in space and its components concerning some basis as the frame of references. Scalars and vectors are special cases of more general objects called tensors of order n with their components. Tensors can be represented as multidimensional arrays of numerical values that transform in a specific way under changes in the system of coordinates and hence can be easily used to represent different physical quantities that can transform in different ways under the coordinate transformation with significant applications and operations.

References

Comon, P. (2014). Tensors: A brief introduction. *IEEE Signal Processing Magazine*, 31(3): 44-53.

Eslami, M. R. (2024). Tensor Analysis. In: *Thermal stresses in plates and shells*, Springer Nature, 7-42.

Goodstein, J. R. (1982). The Italian Mathematician of Relativity. *Centaurus,* 26(3): 241-261.

Guo, H. (2021). *What are Tensors exactly?* World Scientific Publishing Company.

Hamilton, W. R. (2001). On Some Extension of Quaternions. *Philosophical Magazine,* Cambridge University Press, Cambridge.

Hess, S. (2015). *Tensors for physics*. Springer.

Islam, N. (2006). *Tensors and Its Applications.* New Age International P Ltd., India.

Ji, Y., Wang, Q., Li, X., & Liu, J. (2019). A survey on tensor techniques and applications in machine learning. *IEEE Access*, *7*, 162950-162990.