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RESEARCH ARTICLE

# Minimum- Maximum Theorem and Its Application

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## Highlights

- The Minimum-Maximum Theorem is an essential result in mathematical analysis
- Minimum-Maximum Theorem delivers a critical starting point
- If there is any $a \in G$ with $|f(a)| \geq |f(z)|$ for all z $\in G$, then $f$ is constant
- The theorem is crucial in calculus to find absolute extrema of functions

## Abstract

*This paper explains the min-max theorem and how it applies to Schwarz's Lemma and the fundamental theorem of algebra. This Theorem also guarantees the minimum and maximum values of continuous functions across closed intervals. Let $G \subset C$ be a connected open set and $f: G, \rightarrow C$ be analytic. If there is any $\alpha \in G$ with $|f(a)| \geq |f(z)|$ for all $z_{\in} G$, then $f$ is constant. Every polynomial of degree n$\geq 1$ with complex coefficients has a zero in C.*

**Keywords:** *Maximum modulus principle, Fundamental theorem of algebra, Schwarz Lemma*

MSC: 11H55, 30C80

## Introduction

A basic idea in mathematics, the Minimum-Maximum Theorem—also referred to as the Extreme Value Theorem—is important in a number of areas, including calculus, optimization, and real analysis. Essential information on the existence of minimum and maximum values for continuous functions on closed intervals is provided by this theorem. We shall examine the Minimum-Maximum Theorem and its uses in mathematics and other fields in this article.

## Understanding the Minimum-Maximum Theorem

The Minimum-Maximum Theorem states that if a real-valued function f is continuous on a closed interval [a, b], then f reaches both its minimum and maximum values within that interval. In other words, there exist real numbers c and d within the interval [a, b] such that:

(i) f(c) is the minimum value of f on [a, b].

(ii) f(d) is the maximum value of f on [a, b].

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On the closed interval, the function has to be continuous. The interval [a, b] needs to be closed, which implies that its endpoints are included. The variational theorem provides a variation characterization of eigen values of compact Hermitian operators on Hilbert spaces. It is often referred to as the Courant- Fischer-Weylmin-max principle. This might be viewed as the starting point for several findings that are similar. When the operator is non-Hermitian, the theorem provides a similar characterization of the associated singular values [1].

Let A be a $n \times n$ Hermitian matrix with Eigen values $\lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \leq \lambda_n$ then

$$\lambda_k = \max_U\{\max_x\{R_A(x)|x \in U \text{and} x \neq 0\}| \dim(U) = k\} \text{ and}$$

$$\lambda_k = \max_U\{\max_x\{R_A(x)|x \in U \text{ and } x \neq 0\}| \dim(U) = n - k + 1\}$$

when,

$$\lambda_1 \leq R_A(x) \leq \lambda_n, \ x \in C^n\backslash\{0\} \quad (1)$$

and these bounds are attained when x is an eigenvector of the appropriate eigen values.

Also, the simpler formulation for the maximal eigen value $\lambda_n$ is given by:

$\lambda_n = \max\{R_A(x): x \neq 0\}$ . Similarly, the minimal eigen value $\lambda_n$ is given by:

$$\lambda_n = \max\{R_A(x): x \neq 0\} \quad (2)$$

The min-max theorem [2] also holds for self-adjoint operators that are (maybe) unbounded. For an example, unbounded self-adjoint operator occurs for infinite dimensional spaces. The spectrum without separate eigen values of finite multiplicity is known as the essential spectrum. We would like to approximate the eigen values and eigen functions when we have eigen values that are below the fundamental spectrum [3,4].

# Objectives

The objectives of this article are:

- Explain the minimum – maximum theorem.
- Describe its application in the fundamental theorem of algebra, Schwarz's lemma and in other fields.

# Minimum-Maximum Theorem

In complex analysis, this theorem has numerous applications. Additionally, Schwarz's lemma and the fundamental theorem of algebra both make use of it. The Minimum-Maximum Theorem deals with the completeness, continuity, and compactness of real numbers. It is frequently covered in advanced calculus or real analysis classes and can be somewhat complicated. This theorem is proved with the help of the Heine-Borel and Bolzano-Weierstrass theorems.

**Theorem 1. (Maximum Modulus principle) [2,3]**: Let G $\subset C$ be a connected open set and $f: G \to C$ be analytic. If there is any $a \in G$ with $|f(a)| \geq |f(z)|$ for all z $\in G$, then $f$ is constant.

**Proof:**

Let us choose $\delta > 0$, so that D(a, $\delta) \subset G$. set $0 < r < \delta$ and then by Cauchy's integral formula, we have,

$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{z-a}$dz. Now in the case of parameterization,

z = a + $re^{i\theta}$ with 0 ≤ $\theta \leq 2\pi,$ dz = $ire^{i\theta}d\theta$

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})}{ire^{i\theta}}d\theta\frac{1}{2\pi}\int_0^{2\pi}f\left(a+re^{i\theta}\right)d\theta$$

Hence,

$|\,f(a)|\ \leq \frac{1}{2\pi}\int_0^{2\pi}|f\left(a+re^{i\theta}\right)|\,d\theta\beta$°F $\leq \frac{1}{2\pi}\int_0^{2\pi}|f(a)|d\theta = |f(a)|$

Using $|\left(a+re^{i\theta}\right)| \leq |f(a)|\,,\ \forall\,\theta$

Now equality in the inequalities.

Since, the integrand $|f\left(a+re^{i\theta}\right)|$ is a continuous function of $\theta$, this implies

$|\,f\left(a+re^{i\theta}\right)| = |f(a)|$for all $\theta$.

Putting, $\alpha = \arg(f(a))$

Now,

$$|\,f(a) = e^{ia}f(a)$$

$$= \frac{e^{-ia}}{2\pi}\int_0^{2\pi}f\left(a+re^{i\theta}\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}e^{-ia}f\left(a+re^{i\theta}\right)d\theta$$

$$\Re|f(a)| = |f(a)| = \frac{1}{2\pi}\ \Re\int_0^{2\pi}e^{-ia}f\left(a+re^{i\theta}\right)d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\Re\left(e^{i\theta}f(a+re^{i\theta}\right)d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|e^{-ia}f\left(a+re^{i\theta}\right)d\theta$$

Using $\Re w \leq |w|$ for w∈ $\mathbb{C} \leq \frac{1}{2\pi}\int_0^{2\pi}|f\left(a+re^{i\theta}\right)|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|f\ (a)|d\theta$ = | f(a)|

and so, we have equality in all inequalities implying then,

$\Re\left(e^{-ia}f\left(a+re^{i\theta}\right)\right) = \left|e^{-ia}f\left(a+re^{i\theta}\right)\right| = |f(a)|$for all $\theta$

$\Re\left(e^{-ia}f\left(a+re^{i\theta}\right)\right) = 0$ and

$e^{-ia}f\left(a+re^{i\theta}\right) = |f(a)|orf\left(a+re^{i\theta}\right) = e^{i\theta}|f(a)$ (3)

Thus, $f(z)$ is constant for z in the infinite compact set {z: |z-a|=r} of G.

**Corollary:** Suppose R ⊂ $C$ is a closed bounded region. If $f\!: R\ \rightarrow C$ is continuous on R, analytic on the interior of R and not constant, then the maximum value of $|f(z)|$ is attained at a point (or points) on the boundary of R and never at points in the interior of R. Moreover, if we write

$f(x+iy) = u(x,y) +\ iv(x,y)$ (4)

Then the maximum value of u (x, y) attained at a point (or points) on the boundary of R and never at points in the interior of

**Proof:** The first part follows from the fact that a continuous function on a closed bounded set attains a maximum value, and from the maximum modulus principle this value cannot be attained in the interior of R. The second part follows from the observation that the modulus of the function [1],

g (z) = $e^{f(z)}$ $is$ $|g(z)| = e^{it(x,y)}$

# Material and Method

## Fundamental Theorem of Algebra

A foundational finding in mathematics, particularly in the area of complex analysis, is the Fundamental Theorem of Algebra. It asserts that there is always at least one complex root for each non-constant polynomial equation with complex coefficients. Put more simply, this theorem states that there is at least one solution for any polynomial equation with degree larger than zero in the complex number system [5].

This theorem can be simply stated as every non-constant polynomial equationas

P(z) = $a_n z^n + a_{n-1} z^{n-1} + \cdots\cdots + a_n\, z + a_0 = 0$

Where $a_n \neq 0$, and $a_i$ is complex numbers has at least one complex root.

This theorem has several applications in analysis, algebra, and other areas of mathematics and is significant in many other disciplines as well. In many mathematical domains, such as computer science, physics, and engineering, it ensures that solutions to polynomial equations exist [6].

**Theorem 2**. **(Fundamental Theorem of Algebra)** [7]: Every polynomial of degree $n \geq 1$ and assume that p(z) $\neq 0$ for all z ∈ ⊆ $C$.

**Proof.** By Cauchy's integral theorem,

$$\int_{|z|=r} \frac{dz}{zp(z)} = \frac{2\pi i}{p(0)} \neq 0 \qquad (5)$$

where the circle is traversed anti –clockwise. Also, since

$$\left|\int_{|z|=r} \frac{dz}{zp(z)}\right| \leq 2\pi r \times \max_{|z|=r} \frac{1}{|zp(z)|} = \frac{2\pi}{min_{|z|=r}|p(z)|} \to 0 \qquad (6)$$

As r → ∞, which is a contradiction. Hence the theorem is proved.

## Schwarz Lemma

Karl Hermann Amandus Schwarz [8, 9] is the name of complex analysis known as the Schwarz Lemma. It offers vital details on the performance of holomorphic functions on the unit disk in the complex plane. The Riemann Mapping Theorem suggests that every connected proper open subset of the complex plane is not entire plane is called biholomorphically which is equal to the unit disk, is one of the key assumptions in using the Schwarz. Moreover, it uses in a number of mathematical fields, such as mathematical physics and differential geometry [3].

## Theorem 3.

Suppose f be a holomorphic function from the open unit disc$\mathbb{D}$ to itself,F(0) = 0 , and |f( $\mathcal{Z}$)| ≤ 1 for all $\mathcal{Z} \in \mathbb{D}$, satisfying the following minimal Eigen value Lemma [3,10,11].

(i) |f($\mathcal{Z}$ )| ≤ | $\mathcal{Z}$ |,

(ii) |f'(0)| ≤ 1, and

(iii) |f($\mathcal{Z}$ )| = | $\mathcal{Z}$ |, where f ≠ $\mathcal{Z}$ (7)

Equality holds only if f($\mathcal{Z}$) is a linear transformation w = $e^{i\alpha}\,\mathcal{Z}$, where $\alpha$ is real constant.

Proof. Now, f($\mathcal{Z}$) is analytic in disc | $\mathcal{Z}$| < 1. From Taylor's expansion about the origin gives

f($\mathcal{Z}$) = $c_0 + c_1\mathcal{Z} + c_2\mathcal{Z}^2 + \cdots\cdots$

From given f(0) =0 , so $c_n$ = 0.

f( $\mathcal{Z}$) = $c_1\mathcal{Z} + c_2\mathcal{Z}^2 + \cdots\cdots$. Let the function

g($\mathcal{Z}$) = $\frac{f(\mathcal{Z})}{\mathcal{Z}} = c_1 + c_2\mathcal{Z} + \cdots\cdots\ \cdots\cdots$ . In this disc | $\mathcal{Z}$| < 1 and g ($\mathcal{Z}$ ) is analytic. Let $\mathcal{Z}$ = a is an arbitrary point of unit disc and also |a| $< r <$ 1.Now, | f( $\mathcal{Z}$ )| ≤ 1 and having circle | $\mathcal{Z}$| = r then inequality gives

|g($\mathcal{Z}$) | = $\frac{f(\mathcal{Z})}{\mathcal{Z}} \leq \frac{1}{r}$ (8)

From the maximum principle, the inequality (2) also holds in the disc| $Z$ | ≤ r,

|g (a) | = $\left|\frac{f(a)}{a}\right| \leq \frac{1}{r}$

If r tends to 1, we see that

|g (a)| ≤ $\left|\frac{f(a)}{a}\right| \leq 1.$

⇒ |$f$(a)| ≤ | $a$ |.

In specific, |g (0) | = $f'$ (0) | ≤ 1. Since, a is arbitrary, we have

|f($\mathcal{Z}$)| ≤ | $\mathcal{Z}$| (9)

For every $\mathcal{Z}$ , | $\mathcal{Z}$| ≤ 1. If the equality in the Equation (9) holds at a single point. This justifies that [12,13]

|g($\mathcal{Z}$)| attains it's maximum and g($\mathcal{Z}$) may tends to constant. So, | g( $a$)| = 1 holds only if

g ($\mathcal{Z}$) = $\frac{f(\mathcal{Z})}{\mathcal{Z}}$ = $e^{i\alpha}$ where f($\mathcal{Z}$) = $\mathcal{Z}\ e^{i\alpha}$ (10)

# Generalizations

The Minimum-Maximum Theorem can be stretched to higher dimensions and more complex settings, such as:

**Multivariate functions**: A continuous function f (x, y) defined on a compact subset of attains its extrema.

**Non-Euclidean spaces:** Similar results hold for continuous functions on compact metric spaces

## 6. Applications of the Minimum-Maximum Theorem [14, 5, 6]

**Calculus**: The theorem is crucial in calculus when finding absolute extrema (maximum and minimum values) of functions. It provides a rigorous framework for solving problems involving extreme values, which are essential in various mathematical applications.

The minimum-maximum theorem, is a result in linear algebra and functional analysis that characterizes eigenvalues of compact Hermitian operators on Hilbert spaces. The other applications of minimum-maximum theorems:

### *Max-Min value of a matrix*

The Max-Min value of a matrix is designed by taking the minimum entry of each row, and then taking the maximum of those minimum entries. Max-Min and Min-Max values of matrices are often used in engineering and science.

### *Max-Min problems*

Max-Min problems are two-step allocation problems. One side must make a move knowing that the other side will learn to move and counter it optimally.

### *Non-conservative systems*

A non-variational generation of a min-max principle can be used to prove new existence results for nonconservative systems.

**Optimization Problems**: The Minimum-Maximum Theorem is extensively used in optimization problems. Engineers, economists, and scientists employ this theorem to find the optimal solutions for various real-world situations, such as minimizing production costs or maximizing profits.

**Physics and Engineering**: In physics and engineering, the theorem is applied to solve problems related to motion, energy, and design. For example, it can help determine the maximum height reached by a projectile or the minimum amount of material required to construct a certain structure.

**Economics:** Economists use the Minimum-Maximum Theorem to analyze consumer behavior and producer choices. It assists in identifying the equilibrium price and quantity in markets, as well as the optimal allocation of resources.

**Machine Learning**: Machine learning algorithms often involve optimization, and the Minimum-Maximum Theorem can be applied to find the best parameters or weights for a given model. This is particularly relevant in deep learning and neural networks.

**Geometry:** The theorem has geometric applications, such as finding the shortest distance between two points on a curved surface or determining the maximum area enclosed by a given length of fencing.

**Biology:** The maximum growth rate or the least concentration of a material needed for a particular biological activity may be understood by using minimum and maximum derivatives in the analysis of growth rates, population dynamics, and enzyme kinetics in biology. These are but a handful of the several disciplines in which knowledge of rates of change and optimization is crucial, and the applications of minimum and maximum derivatives are numerous.

## Conclusions

The Minimum-Maximum Theorem is an essential result in mathematical analysis, with far-reaching implications in various domains of science and engineering. Its declaration of the existence of extrema for continuous functions simplifies problem-solving and shapes a foundation for more advanced theories. Whether solving optimization problems, analyzing physical systems, or developing numerical algorithms, this theorem delivers a critical starting point.

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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