# On Fundamental Properties in Fuzzy Metric Space

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**Abstract:** *Fuzzy metric spaces theory is important in mathematics, statistics, computer science, and other fields. In this paper fundamental characteristics of the fuzzy metric space are examined. The notions of fuzzy convergent sequence, fuzzy Cauchy sequence, fuzzy open ball, are reviewed with theorems associated with these concepts.*

**Key words :** t-norm , Fuzzy logic, Fuzzy metric space, Fuzzy open ball, Fuzzy convergence,

**Introduction**

Prof. Zadeh introduced fuzzy sets and fuzzy logic in 1965 to deal with the vagueness and uncertainty in mathematics. Since then, there has been a lot of research in the fields of fuzzy logic and fuzzy sets since then. In comparison to classical set theory, fuzzy set theory takes a different approach. Fuzzy logic, in particular, might be understood as an attempt to combine two unique skills. First, the ability to reason and make rational decisions in the face of ambiguity, uncertainty, incompleteness of knowledge, contradictory information, the partiality of truth, and the partiality of possibility. Second, without any measurement or computation, is the ability to do a wide range of physical and mental tasks [Zadeh 2008]. In the discipline of topology, fuzzy logic and fuzzy set theory are commonly used. Fuzzy topology is a key branch of fuzzy theory with a huge research area and a diverse set of applications. Many writers have actively participated in solving difficulties in fuzzy topology to get a suitable definition of fuzzy metric, as time has demanded. Many authors have looked into such issues in fuzzy topology from various perspectives. Karmosil and Michalek [6] created fuzzy metric space in 1975 as a generalization of Menger Space, which is a statistical metric notion. The fuzzy metric was defined by O. Kaleva and S. Seikkala [7] as the distance between two points in a collection expressed in fuzzy real values. Georage and Veeramani [2] adapted the fuzzy metric approach introduced by Karmosil and Michalek in 1994. Hausdorff Space was created for that fuzzy metric space. They've also demonstrated that each metric generates a fuzzy metric. Fuzzy metrics can be used to solve challenges with uncertain and imprecise data. In a fuzzy metric space, A. George Veeramani [10] defined the Hausdorff topology. Every ordinary metric space can produce a fuzzy metric space that is complete whenever the original one does, according to Xia, Z. Q., and Guo, F. F. [11]. With regard to George and Veermani, Gregori, V., Morillas, S., and Sapena, A.[3] offered examples of fuzzy metrics. C. T. Aage, B. S. Choudhury, and K. Das [1] ostak, A.[10] proposed a revised fuzzy metric as an alternative to the concept of a fuzzy metric.

**Definitions and Preliminaries**:

**Distance Function[2]:** Let X be a non-empty set (crisp) and *d* be a function from *d*: X ×X → R+ (non-negative real number) such that for all *x, y, z* ∈ X we have

$M_1$ : $d(x, y) \geq 0$

M2: $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$,

M3: $d(x, y) = d(y, x)$, and

M4: $d(x, y) \leq d(x, z) + d(z, y)$.

A function *d* satisfying the above conditions is said to be a distance function or metric and the pair (X, *d*) is called a metric space.

Example: Let X= $\mathbf{R}^n$ and define a function *d*: X ×X → **R** by the relation

$d(x, y) = \max_{1\leq i\leq n} |x_i - y_i|$

is a metric on X= $\mathbf{R}^n$.

**t – Norm [2]**

Consider a binary operation $*$ : [0, 1] × [0, 1] → [0, 1] satisfying the following conditions :

(i) $1 * a = a$,

(ii) $a * b = b * a$

(iii) $c * d \geq a * b$ whenever $c \geq a$ and $d \geq b$,

(iv) $(a * b) * c) = a * (b * c)$ , for all $a, b, c, d \in [0,1]$

Then the binary operation * is called continuous **t-norm**

**Example:**

i) Let $a * b =$ min( *a, b* ) for all $a, b, \in [0,1]$, then * is a continuous t – norm.

ii) Suppose $a \, o \, b = ab$, for all $a, b, \in [0,1]$ and $ab$ is the usual multiplication in [0, 1 ]. Then o is continuous t-norm on [0,1]

iii) $a * b = \max\{ 0, a + b - 1\}$ then * is continuous t- norm.

iv) $a * b = \begin{cases} a \text{ when } & b = 1 \\ b \text{ when } & a = 1 \end{cases}$ then * is continuous t- norm

Let X be universe of discourse and $A \subseteq X$.

A **fuzzy set [** 12] in X is defined as the collection of order pair $( x , \mu_A)$ where $\mu_A : X \to [0,1]$ and $x \in X$. So that

$A = \{( x , \mu_A(x) ) : x \in X \text{ and } \mu_A : X \to [0,1]\}$

Here, $\mu_A(x)$ is the degree of the element *x* and is called **membership function [** 12]. If $\mu_A(x) = 1$ , then *x* is fully included in the set *A* and if $\mu_A(x) = 0$ means *x* is not included in *A*.

The maximum value of the membership function is known as **height of the fuzzy set [** 12**].** The fuzzy set having height one is called **normalized fuzzy set [** 12**]**. A fuzzy set *A* on X is called **convex** if for all $x_1 , x_2 \in A$ and $\alpha \in [ 0, 1]$

$\mu_A(\alpha x_1 + (1 - \alpha)x_2) \geq \text{minimum} [ \mu_A(x_1) , \mu_A(x_2)]$

**Fuzzy logic [12] is defined as many-valued logic with truth values of variables ranging from 0 to 1.** Fuzzy logic provides a methodology for dealing with linguistic variables. It facilitates common sense reasoning with an imprecise and vague proposition dealing with natural language. It serves as a basis for decision analysis and control activities in the fields of business, finance, and other fields.

A fuzzy set ***A*** is a **fuzzy number[ 12]** if the following conditions are satisfied

- ***A*** is a convex set
- ***A*** is a normalized set

its membership function is piecewise continuous

**Fuzzy Metric Space[6] (Kramosil and Michalek)**

Let X be a non-empty set and $*$ is a t-norm. A fuzzy metric on the set X is a function $p: X \times X \times (0, \infty) \rightarrow [0, 1]$

(KM i) $p(x, y, t) > 0$,

(KM ii) $p(x, y, t) = 1$ for all if and only if $x = y$,

(KM iii) $p(x, y, t) = p(y, x, t)$,

(KM iv) $p(x, z, t + r) \geq p(x, y, t) * p(y, z, r)$

(KM v) $p(x, y, .) : [0, \infty) \rightarrow [0,1]$ is continuous for all $x, y, z \in$ X and $t, r > 0$.

Then the 3- tuple ( X , $p$, $*$ ) is called fuzzy metric space.

**Fuzzy Metric Space[2] (George and Veeramani)**

Let $X$ be a non-empty set and $*$ is a t-norm. A fuzzy metric on the set $X$ is a function $p: X \times X \times (0, \infty) \rightarrow [0, 1]$

(GV i) $p(x, y, t) > 0$,

(GV ii) $p(x, y, t) = 1$ for all if and only if $x = y$,

(GV iii) $p(x, y, t) = p(y, x, t)$,

(GV iv) $p(x, z, t + r) \geq p(x, y, t) * p(y, z, r)$

(GV v) $p(x, y, .) : (0, \infty) \rightarrow [0,1]$ is continuous

for all $x, y, z \in$ X and $t, r > 0$.

Then the 3- tuple ( X , $p$, $*$ ) is called fuzzy metric space.

**Examples[3,4]**

1. Let X be a non-empty set and d be a metric defined on X. Suppose $*$ is continuous t-norm defined by $a * b = a . b$ for all $a, b \in [0, 1]$. Define a relation as
$p(x, y, t) = \frac{t^2}{t^2 + d(x,y)}$ is a fuzzy metric in [ 0, 1] and ( X , $p$, *) is a fuzzy metric space.
2. Let $f: (0, \infty) \rightarrow (0, \infty)$ be an increasing function and d be metric on X. Then
$p(x, y, t) = \frac{f(t)}{f(t) + a.d(x,y)}$ for $a > 0$,
is a fuzzy metric on X and ( X , $p$, .) is fuzzy metric space.
3. Let X is the set of all real numbers and $a * b = a.b$ for all $a, b \in [0, 1]$. Then any function defined by
$$\rho(x, y, t) = \left[ e^{\frac{-|x-y|}{t}} \right] = e^{\frac{-d(x,y)}{t}}$$
for all $x, y \in \boldsymbol{R}$ and $t \in (0, \infty)$

is a fuzzy metric on X = **R**

4. Suppose $X = [0, \infty)$ then the function defined by
$$\rho(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

is a fuzzy metric on X and so ( X, $\rho$ , *) is a fuzzy metric space.

Let ( $X$, p, * ) be a fuzzy metric space and let $x \in X$. Let $r \in (0, 1)$ and $t > 0$, then the **fuzzy open ball** [ 6] with center at $x$ and radius $r$ is denoted by $B(x, r, t)$ and defined by

$B(x,r,t) = \{ y \in X : p(x, y, t) > 1 - r\}$

A sequence $\{x_n\}$ in a fuzzy metric space ( $X, p, *$ ) is said to **converge [ 9 ]** to $x$ if for every $\varepsilon > 0, \exists N$ such that $x_n \in B(x, \varepsilon, t)\ \forall n \geq N$ .

The sequence $\{x_n\}$ in a fuzzy metric space ( $X$, p, * ) is said to be **Cauchy** [9] if

$$\forall \varepsilon > 0 \text{ and } \forall t > 0\ \exists n_o \in Z_+ : p(x_n, x_m, t) > 1 - \varepsilon \quad \forall m, n \geq n_o$$

**Theorem:**

Suppose ( X, p ,* ) be a fuzzy metric space and for $0 < a < 1$, the mapping defined by

$p_a(x, y) = \inf\{t : p(x, y, t) > a\}$ **is** a metric on X

**Proof:** Suppose ( X , p, * ) be a fuzzy metric space and define a relation

$$p_a(x, y) = \inf\{t : p(x, y, t) > a\}$$

Now , we show $p_a$ is a metric on X.

i. We have $p_a(x, y) = \inf\{t : p(x, y, t) > a\}$. Here $p_a(x, y)$ being the infimum of non-negative value, so that $p_a(x, y) > 0$.

ii. $p_a(x, y) = \inf\{t : p(x, y, t) > a\}$ $\inf\{t : p(y, x, t) > a\}$ $= p_a(y, x)$

iii. Suppose that $x = y$ then $p(x, y, t) = 1$ and so we have

$$\{t : p(x, y, t) > a\} = (0, \infty)$$

$$\Rightarrow \inf\{t : p(x, y, t) > a\} = \inf(0, \infty)$$

$$\Rightarrow p_a(x, y) = 0$$

So that if $x = y$ then $p_a(x, y) = 0$.

The converse of the proof is omitted here.

iv. Let $x, y, z \in X$. If $x = y$ or $y = z$ or $x = z$ then we can see that $p_a(x, z) \leq p_a(x, y) + p_a(y, z)$. So we assume that $x \neq y \neq z$.

We have, $p_a(x, y) = \inf\{t : p(x, y, t) > a\}$

$$p_a(y, z) = \inf\{t : p(y, z, t) > a\}$$

$$p_a(x, z) = \inf\{t : p(x, z, t) > a\}$$

So that , $p\big(x, y, p_a(x, y)\big) > a$

and $p\big(y, z, p_a(y, z)\big) > a$

$\Rightarrow p\big(x, y, p_a(x, y)\big) + p\big(y, z, p_a(y, z)\big) > 2a > a$

i.e $p\big(x, y, p_a(x, y)\big) + p\big(y, z, p_a(y, z)\big) > a$

Since, $p$ is a metric, we have

$$p\big(x, z, p_a(x, y) + p_a(y, z)\big) \geq p\big(x, y, p_a(x, y)\big) + p\big(y, z, p_a(y, z)\big) > a$$

i.e $p\big(x, z, p_a(x, y) + p_a(y, z)\big) > a$

$\Rightarrow p_a(x, y) + p_a(y, z) \in \{t : p(x, z, t) > a\}$

$$\Rightarrow p_a(x, y) + p_a(y, z) \geq \inf\{t : p(x, z, t) > a\}$$

$$\Rightarrow p_a(x, y) + p_a(y, z) \geq p_a(x, z)$$

Hence, $p_a(x, y)$ is a metric on X.

Similarly, we can show that , for $0 < b < 1$, the mapping defined by

$p_b(x, y) = \sup\{t : p(x, y, t) > b\}$ is also a metric on X.

**Theorem** [ 2]: Every open ball is an open set in fuzzy metric space**.**

**Theorem [2] :** Every fuzzy metric space is a Hausdroff space

**Theorem:** Let ( $X, p, *$ ) be a fuzzy metric space such that every Cauchy sequence in the space has convergent subsequence , then ( $X, p, *$ ) is complete.

**Proof**: Suppose $\{x_n\}$ be a Cauchy sequence in ( $X, p, *$ ) and $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$ such that $x_{k_n} \to x$. To complete the proof we show that $x_n \to x$. For let $t > 0$ and $0 < \varepsilon < 1$ such that

$(1 - r\ ) * (1 - r\ ) > 1 - \varepsilon$

Here, $\{x_n\}$ is a Cauchy sequence, so for $0 < r < 1$

$$\forall t > 0 \; \exists\, n_o \in Z_+ : p(x_n, x_m, t) > 1 - r \quad \forall m, n \geq n_o$$

Since, $x_{k_n} \to x$ so there $\exists\, k_n \in Z_+$ :

$k_n > n_o \Rightarrow p\left(x_{k_n}, x, \frac{t}{2}\right) > 1 - r$

Now , for $n \geq n_o$ , $p(x_n, x, t) = p\left(x_n, x, \frac{t}{2} + \frac{t}{2}\right)$

$$\geq p\left(x_n, x_{k_n}, \frac{t}{2}\right) * p\left(x_{k_n}, x, \frac{t}{2}\right)$$

$$> (1 - r\ ) * (1 - r\ ) > 1 - \varepsilon$$

Hence $p(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_o$. Hence ( $X, p, *$ ) is complete.

**Theorem:** Let $\{x_n\}$ be a sequence in a fuzzy metric space ( $X, \rho, *$ ). If $\{x_n\}$ converges to $x$ then $x$ is unique limit point of $\{x_n\}$.

**Proof :** Suppose $\{x_n\}$ be a sequence in fuzzy metric space ( $X, \rho, *$ ) such that $x_n \to x$. We claim that $x$ is unique. If possible let us suppose $\{x_n\}$ converges to both $x$ and $y$. Then ,

For, $\varepsilon_1 > 0, \exists\, N_1 \in Z_+ : \rho(x_n, x, t) > 1 - \varepsilon_1$ for all $t > 0$

and , for $\varepsilon_2 > 0, \exists\, N_2 \in Z_+ :$ $\rho(x_n, y, t) > 1 - \varepsilon_2$ for all $t > 0$

Now, let us consider $\varepsilon > 0$ such that $(1 - \varepsilon_1)(1 - \varepsilon_2) > (1 - \varepsilon)$ and N = max $\{N_1, N_2\}$. Then ,

$$\rho(x, y, t + s) \geq \rho(x_n, x, t) * \rho(y_n, y, s)$$

$> (1 - \varepsilon_1)(1 - \varepsilon_2) > (1 - \varepsilon)$

$\therefore \rho(x, y, t + s) > (1 - \varepsilon)$ for all $n \geq N$

$\Rightarrow x = y$. Thus $x$ is unique.

**Theorem:** Let $(X, \rho, *)$ be a fuzzy metric space and suppose $\{x_n\}$ and $\{y_n\}$ be sequences converging to $x$ and $y$ respectively, then $\{x_n + y_n\}$ converges to $x + y$.

**Proof:** Let $(X, \rho, *)$ be a fuzzy metric space and suppose $\{x_n\}$ and $\{y_n\}$ be sequences converging to $x$ and $y$ respectively.

Then for $\varepsilon_1 > 0, \exists\, N_1 \in Z_+ : \forall\, n \geq N_1$, $\rho(x_n, x, t) > 1 - \varepsilon_1$ for all $t > 0$.

Also , for $\varepsilon_2 > 0, \exists\, N_2 \in Z_+ : n \geq N_2$, $\rho(y_n, y, s) > 1 - \varepsilon_2$ for all $s > 0$.

Let, N = max { $N_1$, $N_2$ } and let $\varepsilon > 0$, such that $(1 - \varepsilon_1) * (1 - \varepsilon_2) \geq (1 - \varepsilon)$

Now, $\rho(x_n + y_n, x + y, t + s) \geq$

$(x_n + y_n, x_n + y, s) * (x_n + y, x + y, t)$

$> (1 - \varepsilon_2) * (1 - \varepsilon_1) \geq (1 - \varepsilon)$

Thus , $\rho(x_n + y_n, x + y, t + s) > (1 - \varepsilon)$

Hence, $\{x_n + y_n\}$ converges to $x + y$ .

**Conclusion :** Fuzzy metric space theory is important in mathematics, statistics, computer science, and other fields. In this paper, fundamental characteristics of the fuzzy metric space are examined. The notions of fuzzy convergent sequence, fuzzy Cauchy sequence, and fuzzy open ball are reviewed.

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