# A GENERAL THEOREM FOR BILATERAL GENERATING RELATIONS

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## INTRODUCTION

The purpose of this paper is to establish a general theorem for bilateral generating relations for a class of polynomials satisfying certain kind of Rodrigues' formula. We may consider this as an extension of the general theorem for bilinear function extended by Saran [6] which also includes a theorem on bilateral generating functions for ultraspherical polynomials derived by Chatterjea [4] . We also discuss some particular cases of this theorem.

The generalized polynomial of our interest will be defined by the following generalized Rodrigues' formula.

$$f_n(x, A, k) = [\phi(x)]^n \mu(n) G(x) (x^{k+1} D + A)^n \{g(x)\}$$

This set of polynomials reduces to Srivastava-Singhal polynomials

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where D = d/dx and the parameters a,k,p and r are unrestricted in general, when

$$A = 0, \phi(x) = x^{-k}, \mu(n) = \frac{1}{n!}, G(x) = x^{-\alpha} \exp(px^{s})$$

$$\text{and } g(x) = x^{\alpha} \exp(-px^{s}).$$

We prove the following theorem:

**Theorem** : If there exists

$$f_n(x, A, k) = [\phi(x)]^n \mu(n) G(x) (x^{k+1} D + A)^n \{g(x)\}$$

where g(x), f(x), G(x) [f(x), G(x) _0] are independent of n with A as real parameters and

$$F(x, A, k; t) = \sum_{m=0}^{\infty} a_m t^m f_m(x, A, k)$$

then there exists

$$\frac{G(x) \exp(x^{-A}) \exp\{-\{(x^{-k} - tk\,\phi(x)\}^{-\frac{A}{k}}\}}{[G(x^{-k} - t\phi(x)k)]^{\frac{-1}{k}}}$$

$$F\left(x^{k} - tk\phi(x)^{-\frac{1}{k}}, Ae^{A}, ke^{A}, \frac{ty\,\phi(x)}{\phi[x^{-k} - kt\phi(x)]^{-\frac{1}{k}}}\right)$$

$$\sum_{r=0}^{\infty} b_r(y) \frac{f_r(x,A,k)}{\mu(r)\, r!} t^r$$

*where*

$$b_r(y) = \sum_{m=0}^{r} (-1)^m (-r)_m\, a_m\, \mu(m)\, y^m.$$

**PROOF**

To start with, we assume the relations

$$(2.1) \qquad F(x,A,k;t) = \sum_{m=0}^{\infty} a_m\, t^m\, f_m(x,A,k).$$

and

$$(2.2) \qquad f_n(x,A,k) = [\phi(x)]^n \mu(n) G(x)\, (x^{k+1} D + A)^n \{g(x)\}$$

We replace t by $ty/\phi(x)$ in (2.1) and employ (2.2) in (2.1) to yield

$$(2.3) \qquad F\left(x,A,k;\frac{ty}{\phi(x)}\right) = \sum_{m=0}^{\infty} a_m \frac{(ty)^m}{[\phi(x)]^m} [\phi(x)]^n \mu(m)$$

$$.G(x)\, (x^{k+1} D + A)^m \{g(x)\}.$$

We now multiply both sides of (2.3) by $[G(x)]^{-1}$ and then

operate by $e^{t(x^{k+1}D+A)}$ to arrive at

$$(2.4) \quad e^{t(x^{k+1}D+A)} \, [G(x)]^{-1} \, F\left(x, A, k; \frac{ty}{\phi(x)}\right)$$

$$e^{t(x^{k+1}D+A)} \sum_{m=0}^{\infty} a_m \, \mu(m) \, y^m (x^{k+1} D + A)^m \, \{g(x)\} \, t^m.$$

Since

$$e^{t(x^{k+1}D+A)} \, \phi(x, A, k; t) = \exp(x^{-A}) \exp\{(x^{-k} - tk)^{-\frac{A}{k}}\}$$

$$.\phi\left((x^{-k} - tk)^{-\frac{1}{k}}, \; Ae^{A}, \; ke^{A}, \; t\right)$$

the left member of (2.4) may be obtained in the form

$$(2.5) \quad e^{t(x^{k+1}D+A)} \, [G(x)]^{-1} \, F\left(x, A, k; \frac{ty}{\phi(x)}\right)$$

$$= \exp(x^{-A}) \exp\left\{-(x^{-k} - tk)^{-\frac{A}{k}}\right\} \left[G(x^{-k} - tk)^{-\frac{1}{k}}\right]^{-1}$$

$$F\left((x^{-k} - tk)^{-\frac{1}{k}}, Ae^{A}, ke^{A}; \frac{ty}{\phi(x^{-k} - tk)^{-\frac{1}{k}}}\right)$$

The right member of (2.4) may be worked out as follows:

$$(2.6) \quad e^{t(x^{k+1}D+A)} \sum_{m=0}^{\infty} a_m \, \mu(m) \, y^m \, (x^{k+1} D + A)^m \, \{g(x)\} \, t^m.$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}(x^{k+1}D+A)^r \;\; a_m\,\mu(m)\,y^m\,(x^{k+1}\,D+A)^m\;\{g(x)\}\,t^m.$$

$$\sum_{m=0}^{\infty}\;\sum_{r=0}^{\infty}\;\frac{a_m\;\mu(m)\;y^m}{r!}\;(x^{k+1}\;D+A)^{m+r}\;\{g(x)\}\;t^{m+r}$$

$$\sum_{r=0}^{\infty}\;\sum_{m=0}^{r} a_m\,\mu(m)\;\frac{y^m}{(r-m)!}\;(x^{k+1}\;D+A)^r\;\{g(x)\}\;t^r$$

$$\sum_{r=0}^{\infty}\;\sum_{m=0}^{r}\frac{a_m\;\mu(m)\;y^m}{(r-m)!\;\mu(r)}\;f_r(x,A,k)\;[\phi(x)]^{-r}\;[G(x)]^{-1}\;t^r$$

$$[G(x)]^{-1}\;\sum_{r=0}^{\infty}\;\sum_{m=0}^{r}\;\frac{a_m\;\mu(m)\;y^m}{(r-m)!\;\mu(r)}\;f_r(x,A,k)\left[\frac{t}{\phi(x)}\right]^r$$

We replace t/ϕ(x) by t and then equate (2.5) and (2.6) to get the desired result

$$(2.7)\qquad \frac{G(x)\;\exp(x^{-A})\;\exp\left\{\{\,x^{-k}-tk\,\phi(x)\,\}^{-\frac{A}{k}}\right\}}{G\left((x^{-k}-t\phi(x)k)^{-\frac{1}{k}}\right)}$$

$$F\left(\,[x^{-k}-tk\;\phi(x)]^{-1/k},\;\;Ae^A,\;\;ke^A;\;\frac{ty\;\phi(x)}{\phi\left((x^{-k}-tk\;\phi(x))^{-1/k}\right)}\right)$$

$$\sum_{r=0}^{\infty} b_r(y)\;\frac{f_r(x,A,k)}{\mu(r)\;r!}\;\;t^r$$

$$\text{where}\;\;b_r(y)\;=\;\sum_{m=0}^{r}\;(-1)^m\;(-r)_m\;a_m\;\mu(m)\;y^m.$$

# SPECIAL CASES

**(a)** We set k = -1, ϕ(x) = 1, A = 0, and replace t by -t and y by -y to arrive at the result

$$(3.1) \qquad \frac{G(x)\, H(x-t, ty)}{G(x-t)} = \sum_{r=0}^{\infty} \frac{(-t)^r}{\mu(r) r!}\, b_r^{(1)}(y)\, h_r(x)$$

$$\text{where} \quad h_r(x) = f_r(x,0,-1) = \mu(r)\, G(x)\, D^r\{g(x)\},$$

$$H(x-t, ty) = F(x-t, 0, -1, ty)$$

$$\text{and} \qquad b_r^{(1)}(y) = \sum_{m=0}^{r} (-r)_m\, \mu(m)\, a_m\, y^m.$$

Which is the general theorem extended by Saran [6].

**(b)** We take,

$$A = 0, \; \phi(x) = x^{-k}, \; \mu(n) = 1/n!$$

$$G(x) = x^{-\alpha} \exp(px^s), \; g(x) = x^{\alpha} \exp(-px^s)$$

$$f_n(x, 0, k) = G_n^{\alpha}(x, s, p, k) \text{(Srivastava - Singhal Polynomial [8])}$$

With these substitutions, the relation (2.7) becomes

$$(3.2)\quad \exp\left[px^{s}\left\{(1-(1-tk)^{-\frac{s}{k}}\right\}\right](1-tk)^{-\frac{\alpha}{k}}$$

$$\cdot\,\psi\left(\frac{x}{(1-tk)^{\frac{1}{k}}},\ \frac{y}{(1-tk)}\right)$$

$$= \sum_{r=0}^{\infty} b_r(y)\ G_r^{\alpha}(x,s,p,k)\ t^r$$

$$where\quad b_r(y) = \sum_{m=0}^{r} a_m \binom{r}{m} y^m,$$

$$\psi\left(\frac{x}{(1-tk)^{\frac{1}{k}}},\frac{y}{(1-tk)}\right) = F\left(\frac{x}{(1-tk)^{\frac{1}{k}}},0,k,\frac{y}{(1-tk)}\right)$$

Thus we have obtained the bilateral generating relation for the Srivastava-Singhal polynomial $G_n^{\alpha}(x,s,p,k)$ . Similar result has been deduced earlier by Singhal-Srivastava [7] and by the authors [2].

(c) We set p = s = 1, a = a +1 and use the relation (Carlitz [3]) $Y_r^{a}(x;k) = k^{-a}\, G_r^{a+1}(x,1,1,k)$ where

[ $\alpha > -1$, k is non-zero positive integer ] in (3.2) to obtain the following bilateral generating relation for the biorthogonal polynomial $Y_r^{\alpha}(x;k)$

$$(3.3)\quad \exp[x\{(1-(1-t)^{-\frac{1}{k}}\}]\,(1-t)^{-\frac{\alpha+1}{k}}\, G\left(\frac{x}{(1-t)^{\frac{1}{k}}}, \frac{y}{(1-t)}\right) = \sum_{r=0}^{\infty} b_r(y)\, y_r^{\alpha}(x;k)\, t^r$$

$$\text{where } b_r(y) = \sum_{m=0}^{r} a_m \binom{r}{m} y^m,$$

$$G\left(\frac{x}{(1-t)^{\frac{1}{k}}}, \frac{y}{(1-t)}\right) = F\left(\frac{x}{(1-t)^{\frac{1}{k}}}, 0, k, \frac{y}{(1-t)}\right)$$

(d) For k=1 and $Y_r^{\alpha}(x;1) = L_r^{(\alpha)}(x)$, (3.3) becomes

$$(3.4)\quad (1-t)^{-\alpha}\exp\left(\frac{-tx}{1-t}\right)\lambda\left(\frac{x}{1-t}, \frac{y}{1-t}\right) = \sum_{r=0}^{\infty} b_r(y)\, L_r^{\alpha}(x)\, t^r$$

$$\text{where, } b_r(y) = \sum_{m=0}^{r} a_m \binom{r}{m} y^m,$$

$$\lambda\left(\frac{x}{(1-t)}, \frac{y}{(1-t)}\right) = F\left(\frac{x}{(1-t)}, 0, 1, \frac{y}{(1-t)}\right)$$

It represents a bilateral generating relation for the Laguerre polynomials which has been proved earlier by Al-Salam [1], Singhal-Srivastava [7] and Changdar and Chatterjea [5] by different methods.

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