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ON FATOU TOPOLOGIES OF INEXTENSIBLE RIESZ SPACES

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ABSTRACT

In this paper we shall discuss the general properties of Fatou topologies on Inextensible Riesz Spaces. We find that on a given inextensible Riesz Spaces, there may be no non-trival Fatou topologies but if it has any Hausdorff Fatou topology, it has only one and this has several special properties.

Keywords: fatou - inextensible - riesz spaces - pseudo-norm - maharam measure space - archimedean spaces

INTRODUCTION

The theory of Riesz space, is very rich and its properties have been extensively studied as the center of research activities. A Riesz space due to Hangarian Mathematician Frigyes Riesz (1880- 1956 AD) is a pre-ordered vector Lattice whose per-order is a partial order. Equivalently, it is an ordered vector space for which the order is a Lattice.

In 1975, D.H. Fremlin studied the structure of the locally solid topologies on inextensible Riesz spaces. He subsequently conjectured that his results should hold true for **ρ-** universally complete Riesz spaces.

In 1977, C. D. Aliprantis and O. Burkinshaw proved that indeed Fremlin's result can be generalized to **ρ-** universally complete Riesz spaces and established a number of interesting properties.

The notion of Fatou topology has been discussed in a book edited by C. D. Aliprantis and O. Burkinshaw.

G. Buskes and A. Van Rool: in 2001 have given three approaches to the notion of squares of Riesz spaces as equivalent results.

In 2007, A. R. Khan and Keith Rowlands have established some results specifically related to the topological completion of a Hausdorff locally solid topological lattice group.

Difination

Let X be a set and let τ ba a family of subsets of X. Then τ is called a topology on X if satisfies following properties

(a) Both the empty set and X are element of τ.

(b) Any union of elements of τ is an element of τ

(c) Any intersection of finitely many elements of τ is an element of τ

For Example : An arbitrary non-empty point set X and the set L of all real (finite valued) function on X; the algebraic operations are as usual, ie; L is a real vector space.

$f \leq g$ iff $f(x) \leq g(x) \ \forall \ x \in X,$

Thus L is a Riesz space. All pairs of functions have a least upper bound; foe f(x), g(x) ∈ L, (sup}(f,g)) (x) = max {f(x), g(x)}.

Also, A topological space X, and $\mathcal{L}$ be set of all continuous function on X, the ordering as above then L is a Riesz space.

We start with the following notions:

If X is a probability space, L°(X) a function space has a familiar topology, that of convergence in measure. Similarly, for any set X, R^X is naturally endowed with its product topology. In both cases, the topology is Fatou. We shall also show that under normal circumstances there are no other locally solid topologies.

Lemma (1.1):

Let E be a Riesz space with a Hausdroff Fatou topology τ. Let $A \subseteq E^+$ de directed upwards and bounded for τ. Then A is dominable.

Proof:

Let $x > 0$ in E. then there is a continuous Fatou p seudo-norm ρ on E such that $\in \rho(x) > 0$ (using the fact that τ is defined by the continuous Fatou p seudo-norms). Now A is bounded, so there is an integer k ≥ 1 such that $\rho(k^{-1} z) \leq \frac{1}{2} \in$ for every z ∈ A. Set B={x∧k^{-1} Z:Z ∈ A }. Then B↑ and

$\rho(x) > \frac{1}{2} \in \geq \sup_{u \in B} \rho(u)$

As x is arbitrary, this shows that A is dominable.

Theorem (1.1):

Let M be an inextensible Riesz space and τ , a Fatou topology on M. then τ is Lebesgue, and is complete in the sense that every Cauchy filter has at least one limit. If τ is Hausdorff, it is Levi.

Proof:

We see that a Fatou topology on a Riesz space E is Lebesgue iff every disjoint order bounded sequence in E^+ is convergent to 0. But if (x_n) is any disjoint sequence in M^+, $(n\, x_n)$ is still disjoint, so,

$x = \sup nx_n$

$n \in N$

exists in M; because τ is locally solid; [0,x] is bounded, and $x_n \to 0$ for τ. Thus τ satisfies the condition and is Lebesgue.

Suppose next that τ is Hausdorff. Let $A \subseteq M^+$ be directed upwards and bounded for τ. Then A is dominable because M is inextensible, A is bounded above as A is arbitrary, τ is Levi.

So if τ is Hausdorff, it is a Levi Hausdorff Fatou topology on a Dedekind complete Riesz space M, and is therefore complete by Nakano's theorem. If τ is not Hausdorff, let N be the closure of {0} for τ,

i.e. $N=\{x: x \in M, \rho(x) = 0 \forall$ continuous Fatou p seudo-norm}.

Then N is a band in M, so $M = N \oplus N^d$ (as M is Dedekind complete). N^d is still an inextensible Riesz space and, with the topology induced by τ , is isomorphic, as topogical Riesz space, to the quotient M/N (this is because τ is locally solid). Now τ is a Hausdorff Fatou topology on N^d, therefore complete, so the quotient topology on M/N is complete; which is the same as saying that M is complete under τ.

Lemma (1.2):

Let M be an inextensible Riesz space, and ρ a Riesz p seudo-norm on M.

If $x_n \downarrow 0$ in M, P(xn) →0 If $F \subseteq M$ is a solid linear subspace such that *P* is strictly positive on F, then every disjoint set in F^+ is countable, and F has the countable sup-property.

Proof:

a) Let $\varepsilon > 0$. Then, there is a $\delta > 0$ such that $\rho(\delta x_0) \leq \in$ because part of the definition of a Riesz p-seudo-norm is that

$\lim_{\alpha \to 0} \rho(\alpha x)=0$ for every x.

Now, $X = \sup_{n\in N} n(x_n - \delta x_0)^+$ exists in M.

Again, there must be an $n \geq 1$ such that $P(n^{-1} x) \leq \varepsilon$ so, for any $m \geq n$,

$$\rho(x_m) \leq \rho(x_m - \delta x_0)^+ + \rho(x_m - \delta x_0)$$
$$\leq \rho(m^{-1} x) + \rho(\delta x_0)$$
$$\leq \rho(n^{-1} x) + \in \leq 2 \in$$

As $\in$ is arbitrary, $\rho(x) \to 0$.

b) If (x_n) is any disjoint sequence in F^+, $x = \sup_{n \in N} nx_n$ exists in M, so $\rho(x_n) \leq \rho(n^{-1} x) \to 0$ as $n \to \infty$. Now if $A \subseteq F^+$ is any disjoint set, $\{x : x \in A, \rho(x) \geq \in\}$, must be finite for every $\in > 0$, so (because ρ is strictly positive on F) A is countable. Thus, F has the countable sup-property.

The essential result we are seeking is below; but the following lemma seems a convenient formulation of the underlying idea.

Lemma (1.3):

Let E be a Riesz space with a Hausdorff Fatou topology τ. Let M be the inextensible extension of E, and let θ be any Fatou p seudo-norm on M. Then θ is continuous on E for τ.

Proof:

(a) For any Fatou p seudo-norm *P* on E, set

$$F_\rho = \{z : z \in E, \rho(z) = 0\}$$

$G_\rho = F_\rho^d = \{y : y \in E, |y| \wedge |z| = 0 \forall y, z \leq x\} \uparrow x$ because $G_\rho = F_\rho^d$ and E is Archimedean. (only Archimedean spaces can carry Hausdorff locally solid topologies).

As ρ is Fatou p seudo-norm,

$$\rho(x) = \sup_{u \in A} \rho(u) = \sup\{\rho(y): y \in G_\rho, 0 \leq y \leq x\}$$

(b) Note that, because F is regularly embedded in M, the restriction of θ to E is a Fatou p seudo-norm on E. Consider the support G_θ of θ in E (every disjoint set G_θ^+ is countable). Let A be the set of all pairs (x, ρ) such that ρ is continuous Fatou p seudo-norm on E, $x \in G_p \cap G_,$,and x>0. Let $B \subseteq A$ be maximal subject to the requirement that $x_1 \wedge x_2 = 0$ whenever (x_1, ρ_1) and (x_2, ρ_2) are distinct numbers of B. Then B must be countable. Enumerate it as (x_i, ρ_i).

(c) Let τ_0 be the topology on G_θ induced by the p seudo-norms ρ_i.Then τ_0 is Hausdorff. For suppose that x ∈ G_θ and x ≠ 0. As τ is Hausdorff there is a continuous Fatou p seudo-norm ρ on E such that $\rho(x) \neq 0$. By the remarks in (a) above, there is a y ∈ G_ρ such that $0 \leq y \leq |x|$ and $\rho(y) > 0$. Now $(y, \rho) \in A$, so, by the maximality of B there must be some i such that $y \wedge x_i > 0$. Because x_i belongs to the support of ρ_i.

i.e. $0 < \rho_i(y \wedge x_i) \leq \rho_i(|x|) = \rho_i(x)$.

As x is arbitrary. τ_0 is Hausdorff.

(d) Suppose, if possible, that θ is not continuous for τ. As θ is p seudo-norm, it must be discontinuous at 0, and there is some ∈ > 0 such that {x: x ∈ E, θ (x) ≤ ∈} is not a neighbourhood of 0 for τ. In particular, we must be able to choose, for each n ∈ E and x_n ∈ E such that

$$\sum_{i \leq n} \rho_i(x_n) \leq 4^{-n}, \theta(x_n) > \in.$$

Again using (a) above, we can choose $y_n \in G_\theta$ such that

$$0 \leq y_n \leq |x_n| \text{ and } \theta(y_n) \geq \in;$$

we still have,

$$\sum_{i \leq n} \rho_i(y_n) \leq 4^{-n}.$$

Let A be the set

$$\left\{\sum_{n\leq n}\rho_i y_i : \mathrm{n}\in \mathrm{N}\right\}\subseteq \mathrm{G}_\theta$$

Then A is directed upwards and bounded for τ_0, as

$$\rho_i(2^n y_n)\leq 2^n\rho_i(\mathrm{y}_n)\leq 2^{-n}\forall \mathrm{i}\leq \mathrm{n}$$

Thus, A is dominable in G_θ. But if follows that A is dominable in M. For given u > 0 in M, either $u \wedge z = 0$ for every $z \in G_\theta$ and $(u - z)^+ = u > 0$ for every $z \in A$, or there is a $u_1 \in G_\theta$ such that $u \wedge u_1 > 0$. As E is order-dense in M, there is a $u_2 \in E$ such that $0 < u_2 \leq u \wedge u_1 \leq u$, and now $u_2 \in G_\theta$ So there is $K \in N$ and a $y \in G_\theta$ such that.

$$0 < y \leq (\mathrm{ku}_2 - \mathrm{z})^+ \leq (\mathrm{ku} - \mathrm{z})^+ \forall \mathrm{z}\in \mathrm{A}$$

As u is arbitrary, A is dominable in M.

So, x = supA exists in M. But in this case $\in\leq\theta(\mathrm{y}_n)\leq\theta(2^{-n}\mathrm{x})\forall \mathrm{n}\in \mathrm{N}$, which is impossible. This contradiction shows that θ is continuous for τ.

Let E be a Riesz space with a Hausdorff Fatou topology, and Let M be an inextensible Riesz space with a Fatou topology. Then any order continuous Riesz homomorphism T:E → M is continuous.

Let N be the inextensible extension of E. Then T has an extension to an order continuous Riesz homomorphism $\tilde{\mathrm{T}}: \mathrm{N}\rightarrow \mathrm{M}$. Let ρ be any continuous Fatou p-seudo-norm on M, and

Let, $\theta = \rho\,\tilde{\mathrm{T}}: \mathrm{N}\rightarrow \mathrm{R}^+$. Then θ is a Fatou p seudo-norm on N, so θ is continuous on E. As the topology on M is defined by the continuous Fatou p-seudo-norms, T is continuous.

Lemma (1.4):

Let M be an inextensible Riesz space with a Hausdorff Fatou topology τ, and let E be any Riesz space with a Fatou topology. Then any order continuous increasing liner map T: M → E must be continuous.

Proof:

For let τ_0 be the topology on M with basic neighborhood of 0.

$$\{\mathrm{x}: \mathrm{T}(|\mathrm{x}|)\in \mathrm{U}\},$$

where U runs through the neighborhood of 0 in E. then τ_0 is Fatou, so, applying above properties to the identity map $(M, \tau) \to (M, \tau_0)$, we see that $\tau_0 \subseteq \tau$ and that T is continuous.

Lemma (1.5):

Let M be an inextensible Riesz space. Then there is at most one Hausdorff Fatou topology on M.

Proof:

Immediate from Lemma (1.4).

Example:

If X is any set, the product topology is the Hausdorff Fatou topology on R^+.

If X is any measure space, we may endow $L^\circ(X)$ with the topology τ defined by the p seudo-norm $\rho_u(x) = \rho|x| \wedge u$ where u runs through $L^1(X)^+$; this is Fatou and Lebesgue. When X is semi finite, τ is Hausdorff; so when X is Maharam. τ is the Hausdorff Fatou topology on the inextensible space $L^\circ(X)$.

Let E = C([0, 1]) and let M be the inextensible extension for E, then the only Fatou topology on M is the indiscrete topology, For if ρ is any Fatou p seudo-norm on M, it induces a Lebesgue topology on M, and is therefore an order continuous increasing functional on M^+ as E as regularly embedded in M :$E^+ \to R$ is order-continuous. The argument, although there applied to linear order-continuous increasing functional applies equally well in the present case to show that ρ (1) = 0, where 1 is the unit of E; it follows that ρ is identically zero on E and therefore on M. Thus M has no non-trivial Fatou p seudo–norm and therefore no non-trivial Fatou topology.

Problem:

Is there any inextensible Riesz space which does have a Hausdorff Fatou topology, but is not isomorphic to $L^\circ(x)$ for any Maharam measure space X?

Metrizable spaces:

As an example of the way in which Riesz space properties on an inextensible space are linked with topological properties, we give the following result:

Theorem (1.2):

Let M be an inextensible Riesz space with a Hausdorff Fatou topology τ. Then τ is metrizable iff M has the countable sup property.

Proof:

(a) If M has the countable sup property, it has the stronger property that every disjoint set in M^+ is countable. For M has a weak order unit; now, if $A \subseteq M^+$ is disjoint, then $\{\rho \wedge x : x \in A\}$ is an order bounded disjoint set, so must be countable. But the map $x \to \rho \wedge x : A \to M^+$ is one-to-one, so A is countable. Now we may use the argument to the proof of theorem (1.1), parts (b) and (c), but with M in the place of G_θ, to see that there is a countable family of Fatou p seudo-norms on M defining a Hausdorff topology τ_0 on M, which is of course Fatou and metrizable. But by lemma (1.5) we have $\tau = \tau_0$

(b) On the other hand, if τ is metrizable let (ρ_i) be a sequence of Fatou p seudo-norms on M defining τ.

Define $\rho : M \to R^+$ by

$$\rho(x) = \sum_{i \in N} \min(2^{-i},\ \rho_i(x)) \forall x \in M$$

Then it is easy to see that ρ is strictly positive Fatou-norm on M, so by lemma (1.2) part (b), M has the countable sup-property.

CONCLUSIONS

Every Fatou topology on in inextensible Riesz Space is Lebesgue. The Fatou topology is Housdorff, then it must be a Levi Housdorff Fatou topology on an inextensible Riesz Space.

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