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# THE β-CHANGE BY FINSLER METRIC OF C-REDUCIBLE FINSLER SPACES IN FINSLER GEOMETRY

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## ABSTRACT

This paper considered about the β-Change of Finsler metric L given by L*= f(L, β), where f is any positively homogeneous function of degree one in L and β andobtained the β-Change by Finsler metric of C-reducible Finsler spaces. Also further obtained the condition that a C-reducible Finsler space is transformed to a C-reducible Finsler space by a β-change of Finsler metric.

**Keywords:** β-change, Finsler Metric, C-reducible Finsler Spaces.

## INTRODUCTION

### Finsler space

Suppose that we are given a function $L(x^i,y^i)$ of the line element $(x^i,y^i)$ of a curve defined in R. We shall assume L as a function of class at least $C_5$ in all its 2n arguments. If we define the infinitesimal distance ds between two point $p(x^i)$ and Q $(x^i+dx^i)$ of R by the relation

$$ds = L(x^i,y^i) \tag{1}$$

then the manifold $m^n$equipped with the fundamental function L defining the metric (1) is called a Finsler space (Cartan, 1934; Asanor, 1985; Rund, 1959; Matsumoto, 1992; Matsumoto et al., 1976; Finsler, 1918; Pandey et al. 1997; Kropina, 1961), if $L(x^i,dx^i)$ satisfies the following conditions.

<u>Condition A</u> : The function $L(x^i,y^i)$ is positively homogeneous of degree one in $y^i$i.e.

$$L(x^i,ky^i) = KL(x^i,y^i),\ k > 0 \tag{2}$$

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Condition B: The function $L(x^i,y^i)$ is positive is not all $y^i$ vanish simultaneously i.e.

$$L(x^i, y^i) > 0 \text{ with } \sum_i (yi)^2 \neq 0 \tag{3}$$

Condition C: The quadratic form

$$\dot{\partial}_i \dot{\partial}_j L^2(x,y)\xi^i \xi^j = \frac{\partial^2 L^2(x,y)}{\partial y^i \partial y^j}\xi^i \xi^j \tag{4}$$

is assmed to be positive definite for any variable $\xi^i$

**C-reducible Finsler space**

A Finsler space of dimension n, more than two, is called C-reducible if $C_{ijk}$ is written in the form (Prasad et al. 2013; Prasad et al. 1998; Asanor, 1981; Matsumoto, 1986; Matsumoto et al. 1980).

$$C_{ijk} = \frac{1}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right),$$

Where $C_i = C_{ijk}g^{jk}$ is the torsion vector and $h_{ij}$ is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$.

**Matsumoto Change of Finsler metric**

Let $F^n = \left(M^n, L\right)$ be an n-dimensional Finsler space on a differentiable manifold $M^n$, equipped with the fundamental function L(x,y).

(Shukla et. al. 2012) introduced the transformation of Finsler metric called Matsumoto change of Finslermetric given by

$$\overline{L}\left(x,y\right) = \frac{L^2}{L-B}$$

where $\beta(x,y) = b_i(x)\, y^i$ is a one form on $M^n$

**Exponential Change of Finsler metric**

Let $F^n = (M^n, L)$ be an n-dimensional finsler space on a differential manifold $M^n$, equipped with the fundamental function L(x,y). (Prasad et al. 2013) introduced an exponential change of finsler metric given by

$$\overline{L}\left(x,y\right) = Le^{\beta/L}$$

where $\beta(x,y) = bi(x)y^i$ is a one form on $M^n$

**β-change of Finsler metric:**

Let $F^n = (M^n; L)$ be an $n-$dimensional Finsler space on the differentiable manifold $M^n$ equipped with the fundamental function $L(x,y)$, (Prasad et.al 2013; Synge, 1925; Shibata, 1984; Park et al. 2001) considered the change of Finsler metric given by

$$L^* (x,y) = f(L,\beta), \tag{5}$$

where $f$ is positively homogeneous function of degree one in L and β and βgiven by $\beta(x,y) = b_i(x)y^i$ is a one-form on $M^n$. The Finsler space ($M^n$; $L^*$ ) obtained from $F^n$ by the β-change (5) will be denoted by $F^{*n}$.

The Homogeneity of $f$ in (5) gives

$$Lf_1 + \beta f_2 = f \tag{6}$$

where the subscripts '1' and '2' denote the partial derivatives with respect to $L$ and $\beta$ respectively.

Differentiating (6) with respect to $L$ and $\beta$ respectively, we get

$LF_{11} + \beta f_{12} = 0$ and $Lf_{12} + \beta f_{22} = 0$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{L\beta} = \frac{f_{22}}{L^2}$$

which gives

$$f_{11} = \beta^2 w, f_{22} = L^2 w, f_{12} = -\beta L w$$

where Weierstrass function $w$ is positively homogeneous function of degree - 3 in Land $\beta$. Therefore

$$Lw_1 + \beta w_2 + 3w = 0 \tag{7}$$

Again $w_2$ is positively homogeneous of degree -4 in Land $\beta$, so

$$Lw_{21} + \beta w_{22} + 4w_2 = 0 \tag{8}$$

Throughout the paper we frequently use above equations (6) to (8) without quoting them.

The concept of concurrent vector field has been given by (Matsumoto et al. 1974; Tachibana, 1950), which is defined as follows:

The vector field $b_i$ is said to be a concurrent vector field if

$$(i)\ b_{ij} = -g_{ij} \qquad (ii)\ b_i\Big|_j = 0 \tag{9}$$

where small and long solidus denote the $h-and\ v-covariant$ derivatives respectively.

It has been proved by Matsumoto that $b_i$ and its contravariant components $b^i$ are functions of coordinates alone. Therefore from (9) (ii), we have

$$C_{ijk}{}^{b^i} = 0$$

## FUNDAMENTAL QUANTITIES OF F*ⁿ

To find the relation between fundamental quantities of $F^n and\, F^{*n}$, we use the following results:

$$\dot{\partial}_i\,\beta = b_i, \qquad \dot{\partial}_i\,L = L_i, \qquad \dot{\partial}_j\,l_i = L^{-1}h_{ij} \tag{10}$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and $h_{ij}$ are components of angular metric tensor of $F^n$ given by $h_{ij} = g_{ij} - l_i l_j = L\,\dot{\partial}_i\,\dot{\partial}_j\,L$

The successive differentiation of (1) with respect to $y^i$ and $y^j$ gives:

$$l_i^* = f_1 l_i + f_2 b_i \tag{11}$$

$$h_{ij}^* = \frac{f\,f_1}{L}h_{ij} + f\,L^2 w\,m_i m_j, \tag{12}$$

where $m_i = b_i - \frac{\beta}{L}l_i$

The quantities corresponding to $F^{*n}$ will be denoted by putting star on the top of those quantities.

From (11) and (12) we get the following relations between metric tensors of $F^n$ and $F^{*n}$.

$$g_{ij}^* = \frac{f\,f_1}{L}g_{ij} - \frac{p\beta}{L}l_i l_j + \left(f\,L^2 w + f_2^2\right)b_i b_j + p\left(l_i b_j + l_j b_i\right), \tag{13}$$

where $p = \left(f_1 f_2 - f\,\beta L w\right)$.

The contravariant components of the metric tensor of $F^{*n}$ will be derived from (13) as follows:

$$g^{*ij} = \frac{L}{f\ f_1} g^{ij} + \frac{pL^3}{f^3 f_1 t}\left(\frac{f\beta}{L^2} - \Delta f_2\right) l^i l^j - \frac{L^4 w}{f\ f_1 t} b^i b^j - \frac{pL^2}{f^2 f_1 t}\left(l^i b^j + l^j b^i\right) \quad (14)$$

$$t = f_1 + L^3 w\Delta, \qquad \Delta = b^2 - \frac{\beta^2}{L^2} \quad (15)$$

putting $q = 3f_2 w + f_2 w,$ we find that

$$\text{(a)} \qquad \dot{\partial}_i\, f = \frac{f}{l} l_i + f_2 m_i, \quad (16)$$

$$\text{(b)} \qquad \dot{\partial}_i\, f_1 = -\beta L w m_i,$$

$$\text{(c)} \qquad \dot{\partial}_i\, f_2 = L^2 w\, m_i,$$

$$\text{(d)} \qquad \dot{\partial}_i\, w = -\frac{3w}{L} l_i + w_2\, m_i,$$

$$\text{(e)} \qquad \dot{\partial}_i\, b^2 = -2C_{\cdot\cdot i},$$

$$\text{(f)} \qquad \dot{\partial}_i\, \Delta = -2C_{\cdot\cdot i} - \frac{2\beta}{L^2} m_i,$$

and

$$\text{(a)} \qquad \dot{\partial}_i\, p = \beta L q m_i, \quad (17)$$

$$\text{(b)} \qquad \dot{\partial}_i\, t = -2L^3 w c_{\cdot\cdot t} + \left(L^3 \Delta w_2 - 3\beta L w\right) m_i,$$

$$\text{(c)} \qquad \dot{\partial}_i\, q = -\frac{3q}{L} l_i + \left(4 f_2 w_2 + 3w^2 L^2 + f w_{22}\right) m_i,$$

where, '.' denotes the contraction with $b^i, viz\, C_{\cdot\cdot i} = C_{jki} b^j b^k$

Differentiating (13) with respect to $y^k$, using (10) and (16), we get the following relation between the Cartan's $C - tensors\, (C^*_{ijk} = \frac{1}{2}\dot{\partial}_k\, g^*_{ij}\ and\ C_{ijk} = \frac{1}{2}\dot{\partial}_k\, g_{ij})$:

$$C^*_{ijk} = \frac{f\ f_1}{L} C_{ijk} + \frac{p}{2L}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{qL^2}{2} m_i m_j m_k \quad (18)$$

It is to be noted that

$$m_i l^i = 0, m_i m^i = \Delta = m_i b^i, h_{ij} l^j = 0, h_{ij} m^j =, h_{ij} b^j = m_i \quad (19)$$

where $m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i$

To find $C^{*i}_{jk} = g^{*ih} C^*_{hjk}$ we use (14), (18), (19), we get

$$C^{*i}_{jk} = C^i_{jk} + \frac{p}{2f\, f_1}\left(h_{jk} m^i + h^i_j m_k + h^i_k m_j\right) + \frac{qL^3}{2f\, f_1} m_j m_k m^i$$

$$-\frac{L}{ft} C_{jk} b n^i - \frac{pL\Delta}{2f^2 f_1 t} h_{jk} n^i - \frac{2pL + L^4 \Delta q}{2f^2 f_1 t} m_j m_k n^i, \qquad (20)$$

where $n^i = fL^2 w b^i + p l^i$

We have the following relations corresponding to the vectors with components $n^i$ and $m^i$ :

$$C_{\cdot ijk} m^i = C_{jk}, \qquad C_{ijk} n^i = f\, L^2 w c_{jk}, \qquad m_i m^i = f\, L^2 w \Delta \quad (21)$$

## β-CHANGE OF C- REDUCIBLE FINSLER SPACE

Let $F^n$ be a C-reducible Finsler space. Then (Matsumoto, 1972)

$$C_{hjk} = \frac{1}{n+1}\left(h_{hj} C_k = h_{hk}\, C_j = h_{jk}\, C_i\right) \qquad (22)$$

where $C_k = C_{hjk} g^{hj}$

Using equation (22) in equation (18), we get

$$C^*_{hjk} = \left(p_k h_{hj} + p_j h_{hk} + p_h h_{jk}\right) + \frac{qL^2}{2} m_i m_j m_k \qquad (23)$$

where $$p_k = \frac{ff_1}{L(n+1)} C_k + \frac{p}{2L} m_k \qquad (24)$$

Using equation (12) in equation (23), we get

$$C^*_{hjk} = \frac{L}{f\, f_1}\left(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}\right) + q_h m_j m_k + q_j m_h m_k + q_k m_j m_h \quad (25)$$

where $$q_h = \frac{qL^2}{6} m_h - \frac{L^3 w}{f_1} p_h \qquad (26)$$

Now suppose that the transformation (1) is such that $(n+1)(f_1 w_2 + 3\beta L w^2) m_h = 6 f_1 w C_h$ then $q_h$=0. So equation (25) reduces to

$$C^*_{hjk} = \frac{L}{f\, f_1}(p_k h^*_{hj} + p_j h^*_{hk} + p_h h^*_{jk}) \qquad (27)$$

which will give $\frac{C_k^*}{n+1} = \frac{L}{f\, f_1} p_k, so\, that$

$$C_{njk}^* = \frac{1}{n+1}\left(C_k^* h_{nj}^* + C_j^* h_{nk}^* + C_n^* h_{jk}^*\right) \tag{28}$$

Hence $F^n$ is also a C-reducible. Therefore we have the following:

Theorem (1) Under the β-change of Finsler metric with the condition $(n+1)(f_1 w_2 + 3\beta L w^2) m_n = 6 f_1 w C_n$ the C-reducible Finsler space by β-change of Finslermeticis trans-formed to a C-reducible Finsler space.

In the theorem (1) we have assumed that $(n+1)(f_1 w_2 + 3\beta L w^2) m_n = 6 f_1 w C_n$. However if this condition is not satisfied then a C-reducible Finsler space may not be transformed to a C-reducible Finsler space. In the following we discuss under what condition a C-reducible Finsler space is transformed to a C-reducible Finsler space by β-change of Finsler metric.

In both the spaces $F^n$ and $F^{*n}$ are C-reducible then from (22) and its corresponding equation for $F^{*n}$ we find, on using (18), that

$$\frac{fL^2 w}{t}[(Q_h\, m_j m_k + Q_j\, m_h m_k + Q_k\, m_j m_h) - f_1(C_{\cdot\cdot h} h_{jk} + C_{\cdot\cdot j} h_{hk} + C_{\cdot\cdot k} h_{jh})]$$
$$= \left(\frac{p}{2L} - \frac{ff_1 r}{L(n+1)}\right)(h_{jk} m_h + h_{hj} m_k + h_{hk} m_j) + \left(\frac{pL^2}{2} - 3f\, L^2 wr\right) m_h m_j m_k \tag{29}$$

where $Q_h = C_h - L^3 w C_{\cdot\cdot h}\; and\; r = (n-2)pt + f_1(3p + L^3 q\Delta)$

Thus, we have the following:

Theorem (2) A C-reducible Finsler space is transformed to a C reducible Finsler space by a β-change of Finsler metric if and only if (29) holds.

The condition (29) of theorem (2) is too complicated to study any geometrical concept of Finsler space. So we consider that our βin β-change of Finsler metric is such that $b_i$ is a concurrent vector field so that $C_{\cdot i} = 0$, $C_{\cdot\cdot i} = 0$. Hence equation (29) reduces to

$$fL^2 w\left(C_n m_j m_k + C_j m_n m_k + C_k m_j m_n\right)$$
$$= \left(\frac{p}{2L} + \frac{ff_1 r}{2L}\right)(h_{jk} m_n + h_{nj} m_k + h_{nk} m_j)$$
$$+ \left(\frac{qL^2}{2} - 3f^2 wr\right) m_n m_j m_k \qquad (30)$$

Contracting this equation with $g^{jk}$, we find

$$2f\,L^3 w\Delta C_n = \{(n+1)(p - f\,f_1 r) + (qL^3 - 6f^2 Lwr)\Delta\} m_n \qquad (31)$$

Hence we have the following:

Theorem (3) If a C-reducible Finsler space is transformed to a C-reducibleFinsler space by a concurrent β-change of Finsler metric, then the vector$C_h$ is along the direction of the vector $m_h$.

## CONCLUSION

a) The C-reducible finsler space is transformed to a C-reductibleFinsler space under the β-change of Finsler metric if

  (i) if (n+1) $(f_1 w_2 + 3\beta L w^2) m_h = b f_1 w c_{h.}$

  (ii) if and only if (29) holds

b) If a C-reducible finsler space is transformed to a C-reducible finsler space by a concurrent β-change of finsler metric, then the vector $c_h$ is along the direction of the vector $m_h$

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