On solutions of convection-diffusion equation in fluid

Uddhab Raj Nyaupane*, **Kedar Nath Uprety*** **and Buddhi Prasad Sapkota** **

* Central Department of Mathematics, Tribhuvan, University, Kathmandu, Nepal.
** Ratna Rajyalaxmi Campus, Tribhuvan University, Kathmandu, Nepal.

Abstract: Convection and diffusion are fundamental transport mechanisms that govern the flow of mass, heat or other quantities in various physical, biological and engineering systems. The work overviews theoretical underpinnings, mathematical formulations and numerical methods to analyze the convection-diffusion process in one and two dimensions, both in steady and transient states. We have compared the analytical and numerical results using finite difference and finite volume methods for fluid flow and temperature distribution. The numerical solutions are in agreement with the analytical solution in one dimension. The work compares convection-diffusion with convection-dominating for larger velocities of fluid flow. The solution obtained from FVM is in line with an analytical solution than the solution obtained from FDM. The numerical simulation for the transient flow is also discussed.

Keywords: Convective velocity; Convection-diffusion; Steady state; Transient state.

Introduction

Partial differential equations, including heat, waves, Poisson and the Laplace equation, describe the real-world problems of many physical issues. The convection-diffusion equation describes a physical phenomenon having the two processes of convection and diffusion of particles, energy, or other physical quantities such as heat transfer, the dynamics of fluids and gases and pollution dispersions. Convection, or advection, describes the heat transfer caused by bulk fluid motion. In this process, transportation of matter occurs from one part of a system to another due to random molecular motion, particularly due to randomness, and transport from higher concentration regions to lower concentration regions to reach an equilibrium state of uniform concentration. Examples of diffusion are heat diffusion, molecular diffusion, and Brownian motion. Adolf Fick in 1855, Joseph Fourier in 1822, and Albert Einstein in 1905 developed their mathematical formulations[2]. There are exact solutions to some simple cases described by these partial differential equations in the domain of regular shapes. The numerical methods employ various discretization and solution methods. The most widely used numerical methods are the finite difference (FD), finite element (FE), and finite volume method (FVM)[27]. The finite volume method is the most frequently employed method in computational fluid dynamics to solve convection-diffusion equations. Additionally, this simulates advection-diffusion with processes dominated by convection[10] which is the process of transferring matter caused by the average velocity of all molecules. Let ρ be the density of the fluid, ϕ is the quantity being transported (e.g., temperature, concentration), t be time, x be the spatial coordinate, u be the velocity of the fluid, Γ is the diffusion coefficient, ν is the kinematic viscosity and S_ϕ is the source term. Then, the general transport equation in differential form is given by equation (1) [27]

$$\frac{\partial}{\partial t}(\rho\phi) + \operatorname{div}(\rho\phi u) = \operatorname{div}(\Gamma\nabla\phi) + S_\phi. \tag{1}$$

The first and second terms on the left side are unsteady and the convective term, respectively, while the first and second terms on the right side are diffusion and the source term, respectively. The steady-state convection-diffusion equation is[27]

$$\operatorname{div}(\rho u\phi) = \operatorname{div}(\Gamma\nabla\phi) + S_\phi. \tag{2}$$

The first work on the use of finite differences to solve physical problems was written by Courant, Friedrichs, and Lewy in 1928. This increased interest in FDM during the 1950s, and 1960s as the Lax Equivalence Theorem examined the idea of stability. Allen and Southwell published the first numerical solutions to the convection-diffusion equation in

Author for correspondence: Buddhi Prasad Sapkota, Department of Mathematics, Ratna Rajyalaxmi Campus, Tribhuvan University, Kathmandu, Nepal.

Email: buddi.sapkota@rrlc.tu.edu.np; https://orcid.org/0000-0001-8881-6386

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the 1950s, which developed a research impetus in the 1970s that has persisted to these days[27]. During the 1960s, CFD was used to design, develop, and construct jet engines and airplanes.

A four-step time-splitting approach was devised by Lin R.K. et al.[14] used a 2-dimensional convection-diffusion equation to solve the convection-diffusion equation using a fourth-order approximation. They verified the suggested approach and estimated the second derivative terms using the fourth-order-accurate-centered Methodology. The numerical approaches for governing convection-diffusion problems started in almost 1969. The first time that computer experiments were publicly and openly expressed was in a Scientific American paper (1965) by Harlow and Fromm, and CFD was born [27]. The Russian mathematician Grisha Shishkin was able to use a piecewise uniform mesh in 1990[2]. Miller, O'Riordan, Hegarty, and Farrell, four Irish mathematicians, have been the main proponents of this theory during the 1990s. Aswin, V. S. et al. developed unconditionally stable explicit-implicit schemes for convection-diffusion problems[3]. The advection-diffusion equation was employed to predict the concentration of pollutant transport, according to a study by Johari et al.. Salkuyeh[22] offered a precise solution to the convection-diffusion equation. Analytical solutions to the one-dimensional diffusion equation have been provided by Kumar et al.[25]. Convection-diffusion problems are the most frequently caused by linearizations of Navier-Stokes equations with high Reynolds numbers. Morton (1996) points out that one important and difficult problem is the numerical approximation of partial differential equations with an adequate description of the interaction between convective and diffusive processes. Divergent (conservative) models are the most widely utilized type of convective transport model. Finite element approaches for transient issues, approximations on general polyhedra (the finite volume method), and finite difference schemes on rectangular grids were covered by Churbanov [7]. Gasmi [9] studied the hydrocarbon system used for petroleum reservoir simulations by using the global pressure formulation of Chavent under assumptions like the applicability of Darcy's law; the incompressibility of the porous medium; fluid compressibility; mass transfer between the oil and the gas in negligible gravity. In order to achieve stable discretization in the convection-dominated regime, Bayramov et al.[4] using a time-dependent generalization of the monotone edge-averaged finite element scheme showed that this method is more appropriate for boundary layer problems. Filbet [8] used the finite volume method for a family of nonlinear parabolic equations with non-homogeneous Dirichlet boundary conditions. Numerical results validate the accuracy of the system and highlight its efficiency in maintaining large-time asymptotic behavior. Mirza et al.[15] used the Laplace and Fourier transform with respect to the temporal variable t and the space coordinates x and y to obtain a fundamental solution of the advection-diffusion equation with time-fractional derivatives. For both the fractional/normal diffusion process and the ordinary case of the normal advection–diffusion phenomenon, the particular solution was obtained from the general solution. The convective term was discretized using discrete exterior calculus by Noguez et al. [18] in which the stabilization of discretization is comparable to the finite element method with linear interpolation functions, was achieved through the use of established stabilization techniques like artificial diffusion. To demonstrate numerical convergence, they undertook numerical experiments on basic stationary and transient cases involving domain discretization using coarse and fine simplicial meshes. Kahlaf [11] found the approximate solution to the two-dimensional Laplace equation with the use of numerical techniques. Utilizing two distinct factors (five and nine points) for the Laplace equation and finite elements in regular shape, the finite difference method was found to be the most accurate approach for determining Dirichlet boundary conditions. To discretize the space-fractional derivative and time discretize the equation, Bi Yanan et al.[6] developed a fully discrete finite volume scheme for the two-dimensional space-fractional convection-diffusion equation and the Crank–Nicholson scheme. The study suggests numerical techniques to solve the model partial differential equations. Theoretical analysis was validated by the data. The study uses numerical techniques to solve the model partial differential equations [5]. There are different convection-diffusion equations, their discretization techniques, and methods of solution. In this paper, we have used finite difference, finite volume, and an analytical solution, to check the convergence and compare the results for one- and two-dimensional convection-diffusion equations.

Methodology

Solving the convection-diffusion equation involves finding the distribution of a quantity in a fluid flow system, considering both convection, which is the advective transport due to fluid motion, and diffusion, the spreading or mixing of the quantity due to molecular processes. We discretize the domain into discrete grids or elements, solve the governing equations of fluid flow using initial and boundary conditions, check the convergence, and validate the results using analytical solutions. Results are presented after post-processing with error calculations. Here we have used finite volume and finite difference methods for steady and transient convection-diffusion equations of one- and two-dimensional.

Governing Equations

The governing equation of the general transport equation is defined by the equation (1). The one-dimensional steady-state diffusion equation without source term is given by

$$\frac{d}{dx}(\Gamma\frac{d\phi}{dx}) = 0. \tag{3}$$

A transient one-dimensional heat conduction equation with a source term is

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial(k\frac{\partial T}{\partial x})}{\partial x} + S. \tag{4}$$

with c, T, and k being the specific heat capacity of the material, temperature, and thermal conductivity, respectively. Some of the other forms of the convection-diffusion equations are:
one-dimensional steady convection-diffusion equation

$$\frac{d}{dx}(\rho u\phi) = \frac{d}{dx}(\Gamma\frac{d\phi}{dx}), \tag{5}$$

one-dimensional transient convection-diffusion equation

$$\frac{d}{dt}(\rho\phi) + \frac{d}{dx}(\rho u\phi) = \frac{d}{dx}(\Gamma\frac{d\phi}{dx}), \tag{6}$$

two-dimensional steady convection-diffusion equation

$$\frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial}{\partial y}(\rho u\phi) = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right), \tag{7}$$

u and v are the velocities in X, and Y-directions, respectively, and ϕ may be considered as temperature, concentration, or velocity.
Two-dimensional transient convection-diffusion equation is given by

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right). \tag{8}$$

Discretization

Discretization is a discrete representation of the geometry of the problem that transforms continuous modes of the differential equations into discrete counterparts. It replaces the exact solution with discrete values and converts them into algebraic equations. These equations are derived from the differential equations governing the fluid flow property ϕ. The grid has cells grouped into boundary zones where boundary conditions are applied [23]. To discretize the one-dimensional diffusion equation (3), we take a general nodal point P, and points E and W to the east and west, respectively. The east-side control volume face is denoted by e and the west-side control volume face is denoted by w. As shown in figure (1), the distances between W_P, w_P, P_e, and P_E are represented by δx_{W_P}, δx_{w_P}, δx_{P_e} and δx_{P_E}, respectively.

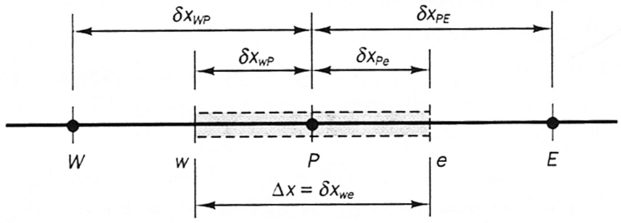

Figure 1: One-dimensional control volume with nodal points E and W, control volume faces e and w and general nodal point P.

We have used finite volume and finite difference discretizations. In the FV method, we get the discretized equation by integrating the governing equation over a control volume, which gives a discretized equation at its nodal point P which yields the control volume. Integration of the steady diffusion equation (3) gives

$$(\Gamma A\frac{\partial\phi}{\partial x})_e - (\Gamma A\frac{\partial\phi}{\partial x})_w = 0 \tag{9}$$

where A is the cross-sectional area of the control volume. Integration with simplification of the convection-diffusion equation (5) gives

$$a_P\phi_P = a_W\phi_W + a_E\phi_E. \tag{10}$$

where a_P, a_W and a_E are the coefficients associated with the P, W and E nodes, respectively. The finite volume discretization equation of the one-dimensional steady-state convection-diffusion equation (5) is given by

$$(\rho uA\phi)_e - (\rho uA\phi)_w = (\Gamma A\frac{\partial\phi}{\partial x})_e - (\Gamma A\frac{\partial\phi}{\partial x})_w. \tag{11}$$

The discretization of the two-dimensional transient heat equation (diffusion equation) using the forward time and central spacing (FTFS) scheme gives

$$\frac{(\phi_{i,j}^{n+1} - \phi_{ij}^{n})}{\Delta t} = \Gamma\left[\frac{(\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n})}{\Delta x^2} + \frac{(\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n})}{\Delta y^2}\right] \tag{12}$$

Rearrange the equation to solve for $\phi_{i,j}^{n+1}$

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \Gamma\Delta t\left[\frac{(\phi_{i+1,j}^{n} - 2\phi_{i,j}^{n} + \phi_{i-1,j}^{n})}{\Delta x^2}\right.$$

$$+\frac{(\phi^n_{i,j+1}-2\phi^n_{i,j}+\phi^n_{i,j-1})}{\Delta y^2}\Bigg] \quad (13)$$

The discretization of the one-dimensional transient convection-diffusion equation using the FTCS for a uniform grid is given by either of the equations (14) or (15).

$$\phi_i^{n+1}=\phi_i^n+\Delta t\frac{\left(u_i(\phi^n_{i+1}-\phi^n_i)-u_{i-1}(\phi^n_i-\phi^n_{i-1})\right)}{\Delta x}$$
$$+\Gamma\frac{\Delta t}{\rho}\frac{\left((\phi^n_{i+1}-\phi^n_i)-(\phi^n_i-\phi^n_{i-1})\right)}{\Delta x^2} \quad (14)$$

$$\phi_i^{n+1}=\phi_i^n+\frac{u\Delta t}{2\Delta x}\left(\phi^n_{i+1}-\phi^n_{i-1}\right)+\frac{\Gamma\Delta t}{(\Delta x)^2}\left(\phi^n_{i+1}-2\phi^n_i+\phi^n_{i-1}\right) \quad (15)$$

for varying and uniform convective velocity respectively.
The discretization of a two-dimensional transient convection-diffusion equation

$$\frac{\partial\phi}{\partial t}+u\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=\Gamma\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) \quad (16)$$

is given by

$$\frac{\phi^{t+\Delta t}_{i,j}-\phi^t_{i,j}}{\Delta t}+u\left(\frac{(\phi^t_{i+1,j}-\phi^t_{i-1,j})}{2\Delta x}+\frac{\phi^t_{i,j+1}-\phi^t_{i,j-1}}{2\Delta y}\right)$$
$$=\Gamma\left(\frac{\phi^t_{i+1,j}-2\phi^t_{i,j}+\phi^t_{i-1,j}}{\Delta x^2}+\frac{(\phi^t_{i,j+1}-2\phi^t_{i,j}+\phi^t_{i,j-1})}{\Delta y^2}\right). \quad (17)$$

Solving equation (16) numerically using the finite volume method needs additional considerations such as time integration methods (e.g., explicit or implicit schemes) and boundary conditions. The choice of grid spacing, time step, and the relationship between the convection and diffusion coefficients can influence the stability and accuracy of the numerical solution. The discretization equation of one-dimensional unsteady heat conduction equation (4) is given by

$$a_PT_P=a_W\left[\theta T_W+(1-\theta)T_W^0\right]+$$
$$a_E\left[\theta T_E+(1-\theta)T_E^0\right]+\left[a_P^0-(1-\theta)a_W-(1-\theta)a_E\right]T_P^0+b \quad (18)$$

where $\theta=0$, and $\theta=1$, the temperature at old and new time levels t and $t+\Delta t$ are used. Finally if $\theta=\frac{1}{2}$, the temperatures at t and $t+\Delta t$ are equally weighted. Here A denotes the control volume's face area, ΔV denotes its volume, which is equal to $A\Delta x$, Δx denotes the control volume's width, and $\overline{S}$ denotes the average source strength. Here $a_P=\theta(a_W+a_E)+a_P^0$ and $a_P^0=\rho u\frac{\Delta x}{\Delta t}$ with $a_W=\frac{k_w}{\delta x_{WP}}, a_E=\frac{k_e}{\delta x_{PE}}$ and $b=\bar{S}\Delta x$.
The exact form of the final discretized equation depends on the value of θ. When θ is zero, we use only temperatures T_P^0, T_W^0 and T_E^0 at the old time level t on the right side of equation (18) to evaluate T_P at the new time, and the resulting scheme is explicit. When $0<\theta\leq 1$ temperatures at the new time level are used on both sides of the equation, and $\theta=\frac{1}{2}$ corresponds to the implicit scheme, while $\theta=1$ gives fully implicit Crank-Nicolson scheme. In the explicit scheme, the source term is linearized as $b=S_u+S_pT_P^0$ and substituting $\theta=0$ into equation (18) gives the explicit form of discretization of the unsteady conductive heat transfer equation

$$a_PT_P=a_WT_W^0+a_ET_E^0+\left[a_P^0-(a_W+a_E-S_P)\right]T_P^0+S_u \quad (19)$$

where $a_P=a_P^0$ and $a_P^0=\rho u\frac{\Delta x}{\Delta t}$.
The right side of the equation (19), the left side can be computed by forward time schemes. The accuracy of the Taylor series truncation error in this approach, which is based on backward differencing, is first-order in terms of time. Thus, the coefficient of T_P^0 may be regarded as the neighboring coefficient that connects the previous values to the present time level, which will be positive if $a_P^0-a_W-a_E>0$. Take k constant and uniform grid spacing, $\delta x_{PE}=\delta x_{WP}=\Delta x$, this condition may be written as

$$\rho u\frac{\Delta x}{\Delta t}>\frac{2k}{\Delta x}. \quad (20)$$

This scheme will only be stable if

$$\Delta t<\rho u\frac{(\Delta x)^2}{2k}. \quad (21)$$

When we set $\theta=\frac{1}{2}$ in equation (18) i.e. the case of Crank - Nicolson scheme, the source term is linearized as $b=S_u+\frac{1}{2}S_PT_P+\frac{1}{2}S_PT_P^0$. So that the discretized unsteady heat conduction equation is

$$a_PT_P=a_E\left[\frac{T_E+T_E^0}{2}\right]+a_W\left[\frac{T_W+T_W^0}{2}\right]$$
$$+\left[a_P^0-\frac{a_E}{2}-\frac{a_W}{2}\right]T_P^0+S_u+\frac{1}{2}S_PT_P^0 \quad (22)$$

where $a_P=\frac{1}{2}(a_W+a_E)+a_P^0-\frac{1}{2}S_P$ and $a_P^0=\rho c\frac{\Delta x}{\Delta t}$.
The method is implicit, based on central differencing, and second-order in time, where all coefficients are positive for physically realistic and bounded results. Here, the time step limitation is only slightly less restrictive than (21) that of the explicit method. For some combinations of Δt and Δx this scheme may also yield physically unrealistic results. When we take $\theta=1$, we get the fully implicit scheme. The source term is linearized as $b=S_u+S_PT_P$. The discretized

equation becomes

$$a_P T_P = a_W T_W + a_E T_E + a_P^0 T_P^0 + S_u. \quad (23)$$

where $a_P = a_P^0 + a_W + a_E - S_P$ and $a_P^0 = \rho u \frac{\Delta x}{\Delta t}$. The three-dimensional convection-diffusion process can be seen in different real-world situations. The solutions to the above discretized equations are presented in the succeeding section.

Results and Discussion

The numerical solutions of the convection-diffusion equation are typically assessed by contrasting them with analytical solutions. The analytical solution provided for a finite domain is the most commonly used one [16]. Here we present some of the numerical solutions of one- and two-dimensional convection-diffusion equations and one- and two-dimensional diffusion equations plotted using Excel and Python.

One-dimensional steady-state heat conduction in an insulated rod without source term is governed by the equation

$$\frac{d}{dx}\left(\kappa \frac{dT}{dx}\right) = 0. \quad (24)$$

κ being the thermal conductivity and T, the dependent variable, and its ends are kept at constant temperatures of $T_A = 100^o C$ and $T_B = 300^o C$ respectively. We use the finite volume approach with thermal conductivity $\kappa = 1000 W/mK$, the cross-section area $0.01 m^2$, and divide the length of the rod into five control volumes. Given the boundary temperatures (T = 400x + 100), the analytical solution is a linear distribution between them. The discretized form of equation (24) using finite difference approximation can be written as

$$T_{i-1} - 2T_i + T_{i+1} = 0$$

After solving above equation, we get $T_1 = 133.33$, $T_2 = 166.66$, $T_3 = 200$, $T_4 = 233.33$, and $T_5 = 266.66$. The solution suggests that the finite volume method is in good agreement with the analytical solution, as shown in figure (2), but the finite difference method has a slight error in an acceptable range.

For one-dimensional steady-state convection-diffusion (5) with ϕ_0 = 1 at $x = 0$ and ϕ_L = 0 at $x = L$, we use seven equally distributed cells and the central differencing discretization for convection-diffusion, where ϕ, a function of x, is calculated for different values $u = 0.1, 1.0, 2.0$ and 2.5 m/s with 20 grid nodes. Taking $L = 1m$, $\rho = 1kg/m^3$and $\Gamma = 0.1kg/m$, the analytical solution of equation (5) between two points E and W is subject to the boundary conditions

$$\phi(x = 0) = \phi_0 \text{ and } \phi(x = L) = \phi_L \quad (25)$$

is given by [17]

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{e^{\frac{x}{L}P} - 1}{e^P - 1} \quad (26)$$

where P is the Peclet number defined by

$$P = \frac{\rho u L}{\Gamma}. \quad (27)$$

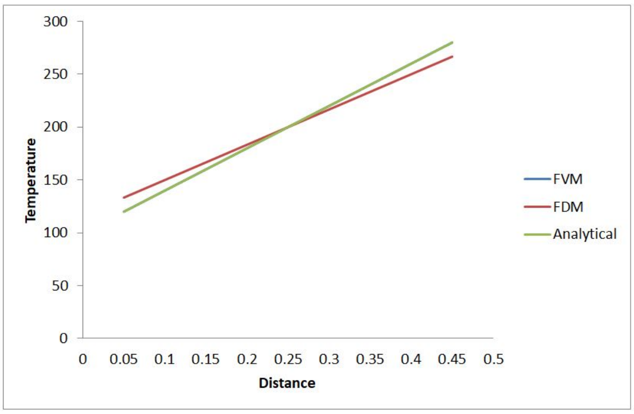

Figure 2: Comparison of FVM and FDM with the analytical solution of the one-dimensional steady-state diffusion equation, where FVM coincides with the analytical solution and FDM has a slight variation.

The discretized equation is

$$a_p \phi_p = a_W \phi_W + a_E \phi_E \quad (28)$$

where a_p, a_W and a_E are the coefficients of ϕ_p, ϕ_W and ϕ_E respectively and applied at internal nodal points 2, 3, 4, 5 and 6, but control volumes (1) and different treatments are used for (7).

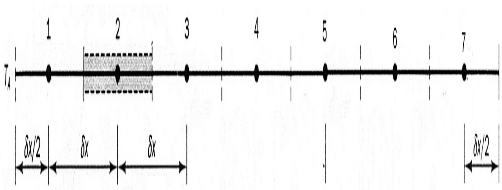

Figure 3: One-dimensional control volume with seven grid points

The discretized equation at its boundary nodes is given by

$$a_p \phi_p = a_W \phi_W + a_E \phi_E + S_u \quad (29)$$

with coefficients

$$a_p = a_W + a_E + (F_e - F_w) - S_p.$$

When velocity $u = 0.1 m/s$, $F = \rho u = 0.1$, $D = \frac{\Gamma}{\delta x} = \frac{0.1}{0.14} = 0.71$, the solution is $\phi_1 = 0.9581$, $\phi_2 = 0.8619$, $\phi_3 = 0.7510$, $\phi_4 = 0.6234$, $\phi_5 = 0.4764$, $\phi_6 = 0.3071$, $\phi_7 = 0.1122$. In the case of the convection-diffusion equation, solutions from FVM are more accurate than those from FDM as shown in figure 4 below for $u = 0.1m/s$. This demonstrates the greater accuracy of the finite volume method in capturing the behavior of convection-diffusion processes at lower velocities.

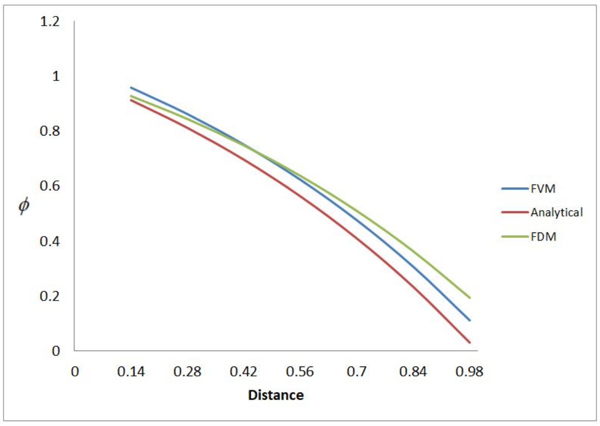

Figure 4: Comparison with an analytical solution with u = 0.1 m/s. The finite volume method is closer to the analytical solution than FDM and the solution converges.

When velocity $u = 1.0m/s$, $F = \rho u = 1$, $D = \frac{\Gamma}{\delta x} = \frac{0.1}{0.14} = 0.71$, the solution is $\phi_1 = 0.9999$, $\phi_2 = 0.9997$, $\phi_3 = 0.9997$, $\phi_4 = 0.99845$, $\phi_5 = 0.9910$, $\phi_6 = 0.9486$, $\phi_7 = 0.70422$. When velocity $u = 1.0m/s$, then the solution by FVM is closer to the analytical solution than FDM. As the distance increases, the solution goes far from the analytical solution, as shown in figure 5.

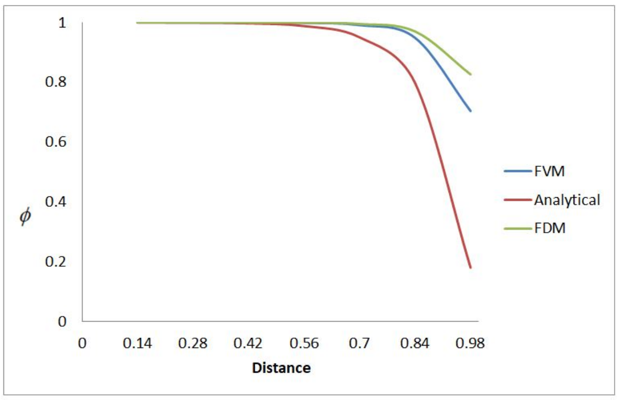

Figure 5: Comparison with the analytical solution of one-dimensional convection-diffusion equation when $u = 1.0m/s$ where the FVM is closer to the analytical solution.

The simulated results show that the finite volume method converges toward an analytical solution faster than the finite difference method.

Low-velocity flow is diffusion-dominated, resulting in smooth and stable numerical solutions, whereas higher-velocity flow becomes convection-dominated. At very high velocities, numerical solutions can diverge, particularly at higher grid points. This necessitates more refined numerical techniques to handle sharp gradients and prevent instability. When velocity $u = 2.0m/s$, $F = \rho u = 2.0, D = \frac{\Gamma}{\delta x} = \frac{0.1}{0.1428} = 0.7142$ (figure 6), the solution is $\phi_1 = 1.0000$, $\phi_2 = 0.9999$, $\phi_3 = 0.9999$, $\phi_4 = 0.9980$, $\phi_5 = 1.01174$, $\phi_6 = 0.93072$, $\phi_7 = 1.40844$.

Again, when velocity $u = 2.5m/s$, $F = \rho u = 2.5, D = \frac{\Gamma}{\delta x} = \frac{0.1}{0.14} = 0.71$, the solution is $\phi_1 = 1.00021$, $\phi_2 = 0.9986$, $\phi_3 = 1.00426$, $\phi_4 = 0.9839$, $\phi_5 = 1.05760$, $\phi_6 = 0.79036$, $\phi_7 = 1.76035$. In cases of higher velocities than $u = 2.0m/s$, the solution diverges due to convection-dominated flow. If the increased nodes satisfied the CFL condition, it converged in figure 7.

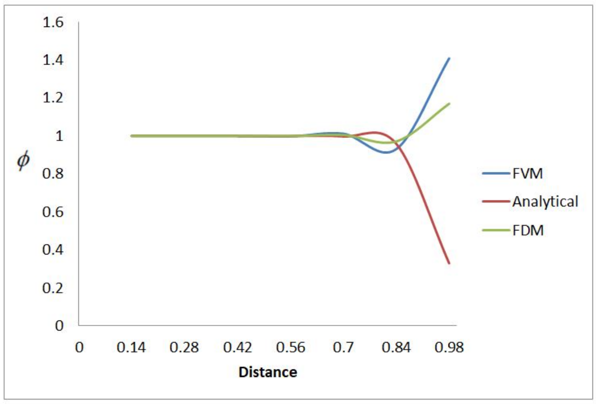

Figure 6: Comparison with the analytical solution of the one-dimensional convection-diffusion equation when $u = 2.0m/s$. For higher velocity, the solution diverges, indicating that the flow becomes more advection-dominated.

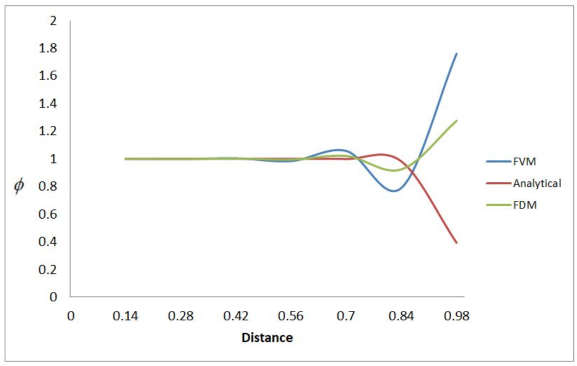

Figure 7: Comparison with the analytical solution of the one-dimensional convection-diffusion equation when $u = 2.5m/s$, case of divergence.

The figure 8 represents the two-dimensional convection-diffusion that describes the transport of a quantity, typically represented by the variable ϕ, in a fluid medium subject to both convection and diffusion. The initial solution is the circular region in the center of the domain, representing the initial conditions.

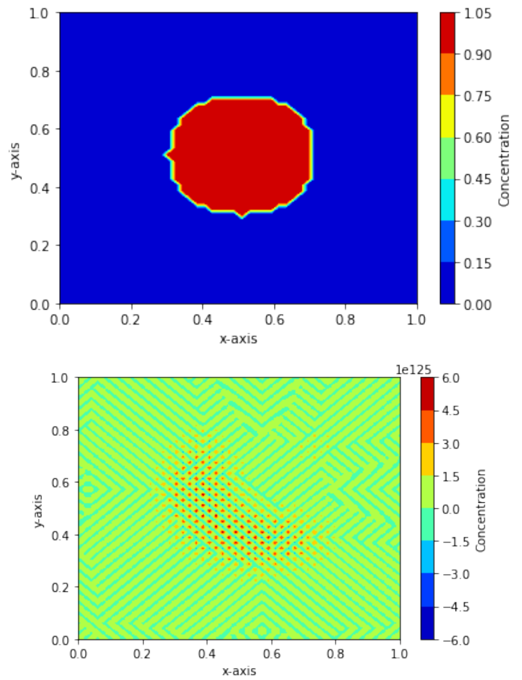

Figure 8: Two-dimensional convection-diffusion equation using the initial solution (left), where the circular region in the center of the domain represents the initial conditions and a transient solution (right), where the convection-diffusion equation drives the evolution of variable ϕ across the domain.

At the beginning of the process, the variable ϕ has a certain distribution within the circular region, and the rest of the domain is initially at some other state. Together, these processes govern how ϕ changes and spread throughout the domain, leading to the evolution of the system over time 10 seconds with lengths in x and y axes taken 1, the number of grid points in x and y axes is 50 each; time steps 1000, c = 1 and $\Gamma = 0.1$.

Conclusion

The convection-diffusion equation is a fundamental partial differential equation used to describe the transport phenomena of fluid flow that combines the effects of convection, the transport of fluid due to bulk movement, and diffusion, the spreading of fluid particles due to random motion. The convection represents the transport of the quantity due to the fluid's bulk motion. Diffusion accounts for the spreading due to molecular or turbulent diffusion. Solving the convection-diffusion equation analytically is often challenging, especially for complex flow fields and boundary conditions. Numerical methods like the finite volume method (FVM) and finite difference method (FDM) are commonly employed to obtain approximate solutions. For one-dimensional convection-diffusion, the finite volume method shows better agreement with the analytical solution. The accuracy and convergence of solutions for steady and transient states are compared. It is observed that the finite volume method provides a more accurate solution than the finite difference method. The numerical solutions converge to the analytical solution for velocities ranging from 0.1 m/s to below $2m/s$, whereas for velocities exceeding 2.0 m/s, divergence occurs at the final grid node. Increasing the number of nodes improves the convergence of the numerical solution, subject to satisfying the Courant-Friedrichs-Lewy conditions. The flow is convection-dominated for higher velocities, while for lower velocities, it is diffusion-dominated.

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