DIRAC PARTICLES IN COULOMB LIKE FIELD IN FLRW–SPACE

S. K. Sharma*, P. R. Dhungel, and U. Khanal*****

*B.P. Koirala Memorial Planetarium, Observatory and Science Museum Development Board, MoEST, Kathmandu, Nepal.

**Department of Physics, St. Xavier's College, Tribhuvan University, Kathmandu, Nepal.

*** Ratobangala School, Lalitpur, Nepal.

Abstract: Behaviour of the Dirac particle in Coulomb like field in FLRW space is investigated. Firstly, the Maxwell equations, in terms of the vector potentials are solved to identify the Lorentz and Coulomb like gauges. The radial Coulomb like potential is solved in terms of Legendre functions. Then the Dirac equation is generalized to include this potential and the angular part is separated and solved. The radial and temporal parts of the massless case is also separated and solved. But the massive case remains coupled. This is still reduced to the case where the Dirac particle can be represented as being in a combined gravitational and electric potential. This effective potential is found to develop an attractive well, which may require a revisit to the recombination era.

Key words: FLRW spacetime; Dirac field; Coulomb like potential; NP formalism.

INTRODUCTION

All matter being ultimately fermions, study of Dirac field is very important to understand the nature and behaviour of the constituents of the universe and its evolutions. Such study becomes much more convenient to formulate in the powerful Newman–Penrose (NP) formalism[1] compared to the conventional tensor method[2,3]. Although NP formalism was primarily used to study quantum fields in black hole space times, we extended its application to write and solve Maxwell, Dirac[4,5] and gravitational perturbation equations[6] in FLRW space-time.

In this work, we use the same formalism and notations as in Chandrasekhar[7] to study the behaviour of the Dirac particles in Coulomb like gauge field in FLRW spacetime. In the first part of the next section, we set up the preliminaries to write the Maxwell's equations in terms of the gauge potentials to identify the Lorentz and Coulomb like gauges, and solve for the potential due to a point charge. In the remaining part of that section, the Dirac equations have been solved to obtain the Coulomb like potentials and force fields. In the third section, some discussions along with concluding remarks have been made.

LORENTZ LIKE GAUGE AND DIRAC FIELD IN COULOMB LIKE POTENTIAL

Lorentz like gauge

In our chosen tetrad frame[1,4,7] for the FLRW spacetime $l_\mu = [1,-1,\ 0,0]$, $n_\mu = \frac{a^2}{2}[1,1,\ 0,0]$, $m_\mu = -\frac{aS}{\sqrt{2}}[0,0,1,i\sin\theta]$ and the complex conjugate $\bar{m}_\mu$, the electromagnetic 4–vector potentials are given by $A_l = A_\mu l^\mu = \frac{1}{a^2}\left(A_\eta + A_r\right)$, $A_n = \frac{1}{2}\left(A_\eta - A_r\right)$, $A_m = \frac{1}{\sqrt{2}aS}\left(A_\theta + \frac{i}{\sin\theta}A_\phi\right)$ and $A_{\bar{m}}$, where

$$S = \frac{\sqrt{k}r}{\sqrt{k}} = \begin{cases} \sin r, & k = 1 \ for\ closed \\ r, & k = 0 \ for\ flat \\ \sinh r, & k = -1 \ for\ open \end{cases} \quad \ldots(1)$$

The Maxwell scalar representing the field strengths are then,

$$F_{\mu\nu} l^\mu m^\nu = \varphi_0 = \frac{1}{a^3 S}\mathcal{D}^{-} aSA_m + \frac{1}{\sqrt{2}aS}\mathcal{L}_0^{-} A_l \quad \ldots(2)$$

Author for correspondence: P. R. Dhungel, Department of Physics, St. Xavier's College, Tribhuvan University, Kathmandu, Nepal.
Email: premrd@hotmail.com

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$$\frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu) = \varphi_1$$

$$= -\frac{1}{2a}\left(\mathcal{D}^+ a^2 \frac{A_l}{2} + \mathcal{D}^- A_n\right) + \frac{1}{2\sqrt{2}aS}(\mathcal{L}_1^- A_{\bar{m}} - \mathcal{L}_1^+ A_m) \ldots(3)$$

and,

$$F_{\mu\nu}\bar{m}^\mu n^\nu = \varphi_2 = \frac{1}{2aS}\mathcal{D}^+ aSA_{\bar{m}} - \frac{1}{\sqrt{2}aS}\mathcal{L}_0^- A_n \ldots(4)$$

We will be using the operator $\mathcal{D}_n^\pm = \left(\frac{\partial}{\partial r} \mp \frac{\partial}{\partial \eta} + n\frac{S\prime}{S}\right)$ and $\mathcal{L}_n^\pm = \left(\frac{\partial}{\partial \theta} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi} + n\cot\theta\right)$. The number of factors of l^μ minus those of n^μ is the boost weight and that of m^μ minus $\bar{m}^\mu$ is spin weight. So, φ_0 is a field of helicity +1, φ_2 of –1 and φ_1 of zero. Substituting these expressions for φ's in the Maxwell equations leads to

$$\mathcal{D}^-\left[S^2\left(\mathcal{D}^+\frac{a^2}{2}A_l + \mathcal{D}^- A_n\right) - \frac{aS}{\sqrt{2}}(\mathcal{L}_1^- A_{\bar{m}} + \mathcal{L}_1^+ A_m)\right]$$
$$+2\mathcal{L}_1^+\mathcal{L}_0^-\frac{a^2}{2}A_l = -a^4S^2J_l \qquad \ldots(5a)$$

$$\mathcal{D}^+\left[S^2\left(\mathcal{D}^+\frac{a^2}{2}A_l + \mathcal{D}^- A_n\right) + \frac{aS}{\sqrt{2}}(\mathcal{L}_1^- A_{\bar{m}} + \mathcal{L}_1^+ A_m)\right]$$
$$+2\mathcal{L}_1^-\mathcal{L}_0^- A_n = -2a^2S^2J_n \qquad \ldots(5b)$$

$$S^2\mathcal{D}^\pm\mathcal{D}^\mp aS\binom{A_m}{A_{\bar{m}}} +$$
$$\frac{\mathcal{L}_0^\mp}{\sqrt{2}}\left[S^2\left(\mathcal{D}^- A_n - \mathcal{D}^+\frac{a^2}{2}A_l\right) \mp \frac{aS}{\sqrt{2}}(\mathcal{L}_1^- A_{\bar{m}} - \mathcal{L}_1^+ A_m)\right]$$
$$= -a^3S^3\binom{J_m}{J_{\bar{m}}} \qquad \ldots(5c)$$

A Lorentz like gauge can immediately be identified as

$$S^2\left(-\mathcal{D}^- A_n + \mathcal{D}^+\frac{a^2}{2}A_l\right) + \frac{aS}{\sqrt{2}}(\mathcal{L}_1^- A_{\bar{m}} + \mathcal{L}_1^+ A_m) = 0 \quad \ldots(6)$$

Whence the Maxwell Eqs. (5a, b, c) become

$$[\mathcal{D}^\pm S^2\mathcal{D}^\mp + \mathcal{L}_0^\mp\mathcal{L}_1^\pm]\binom{a^2A_l}{A_n} = -a^2S^2\binom{J_l}{J_n}$$

and

$$[S^2\mathcal{D}^\mp\mathcal{D}^\pm + \mathcal{L}_0^\mp\mathcal{L}_1^\pm]aS\binom{A_m}{A_{\bar{m}}} = -a^3S^3\binom{J_m}{J_{\bar{m}}} \ldots(7)$$

We call (6) Lorentz like condition. It is the vanishing of the 4–divergence of $\frac{A^\mu}{(aS)^2}$ rather than that of A^μ.

We have already used the s–spin weighted spherical harmonics ${}_sY_l^m$ that obey the spin lowering (raising) operations $\mathcal{L}_s^\pm\ {}_{\pm s}Y_l^m = \pm L_{s-1\ \pm(s-1)}Y_l^m$ where $L_s^2 = (l-s)(l+s+1)$ and the s–boost weight functions ${}_sZ_k^\omega$ obeying $S\mathcal{D}_{-s}^\pm\ {}_{\pm s}Z_k^\omega = \pm iK_{s-1\ \pm(s-1)}Z_k^\omega$ where $K_s^2 = (k-s)(k+s+1)$.

We also identified ${}_pY_l^m = \frac{N_1}{\sqrt{2\pi}}e^{im\varphi}(1-\cos\theta)^{\frac{m+p}{2}}(1+\cos\theta)^{\frac{m-p}{2}}P_{l-m}^{(m+p,m-p)}(\cos\theta)$ as spherical harmonics formed with Jacobi polynomials[8] $P_n^{(\alpha,\beta)}$, and ${}_pZ_k^\omega = \frac{N_2}{\sqrt{2\pi}}e^{-im\eta}(1-i\cot r)^{-\frac{\omega+p}{2}}(1+i\cot r)^{-\frac{\omega-p}{2}}P_{\omega-k-1}^{(-\omega-p,-\omega+p)}(i\cot r)$. So, we can just read off the solutions of (7) as

$$\binom{a^2A_l}{A_n} \sim \frac{1}{S}\ {}_{\pm1}Z_k^\omega\ {}_0Y_l^m \text{ and } aS\binom{A_m}{A_{\bar{m}}} \sim {}_0Z_k^\omega\ {}_{\pm1}Y_l^m.$$

For solutions of homogenous equations it is just the condition $k = l$.

For Coulomb like gauge we add the equations (5a) and (5b) and find that the condition

$$\frac{\partial}{\partial r}S^2\left(\frac{a^2}{2}A_l - A_n\right) + \frac{aS}{\sqrt{2}}(\mathcal{L}_1^- A_{\bar{m}} + \mathcal{L}_1^+ A_m) = 0 \quad \ldots(8)$$

gives

$$\left[\frac{\partial}{\partial r}S^2\frac{\partial}{\partial r} + \mathcal{L}_1^+\mathcal{L}_0^-\right]A_\eta = -na^3S^2 = -n_0a_0^3S^2 \ldots(9)$$

The potential of a point charge q located at the origin is found to be

$$A_\eta = \frac{q}{4\pi}\frac{S'}{S} == \frac{q}{4\pi}\begin{cases}\cot r, & k=1\\ \frac{1}{r}, & k=0\\ \coth r, & k=-1\end{cases}$$

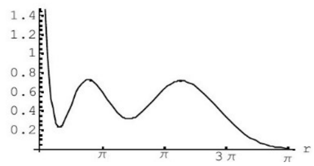
We may proceed to solve the general equation (9) for the Green's function by standard techniques[9] to find (for the closed Universe with $k = 1$)

$$G(r,r') =$$
$$\frac{\pi}{2}\frac{1}{\sqrt{\sin r\ \sin r\prime}}\sum_{l=0}^\infty \frac{(-l)(2l+1)}{4\pi}P_{\frac{1}{2}}^{-\left(l+\frac{1}{2}\right)}(\cos r')\,P_{\frac{1}{2}}^{\left(l+\frac{1}{2}\right)}(\cos r)P_l\cos\beta \ \ldots(10)$$

for $r > r'$ and rr' for $r < r'$. We expect that an addition formula for Associated Legendre functions $P_{\frac{1}{2}}^{\pm\left(l+\frac{1}{2}\right)}(\cos r)$ give

$$G(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi}\cot\rho \quad \ldots(11)$$

where $\cos\rho = \cos r\cos r' + \sin r\sin r'\cos\beta$



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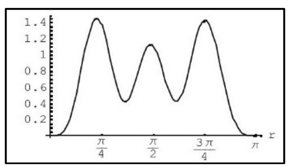

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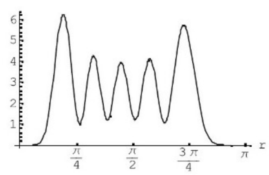


(c)

Figure 1: ***Plot of*** $|R_+|^2 + |R_-|^2$ ***representing the relative probability against r for αz = 1. (a) for n = 2 &*** $l = 1/2$, ***(b) for n = 2 &*** $l = 3/2$, ***(c) for n = 4 &*** $l = 9/2$.

Dirac field in a Coulomb-like potential

Next we use the previous result to look at the analogue of the hydrogen like atom in an expanding universe. We have dealt with the Dirac equation in another work[4] and the radial part is given by the equation

$$\mathcal{D}^{\pm}\Phi_{\pm} = -\left(\frac{\lambda}{S} \pm iMa\right)\Phi_{\mp} \quad \ldots(12)$$

where $\lambda = l + \frac{1}{2}$ and $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots\ldots$

We have just changed the sign of the eigenvalue λ of eigenfunction $\mathcal{L}_{\frac{1}{2}}^{\pm}\ {}_{\pm\frac{1}{2}}Y_l^m = \pm\left(l+\frac{1}{2}\right){}_{\mp\frac{1}{2}}Y_l^m$ to be consistent with other ${}_sY_l^m$.

We just make the minimal substitution, $\partial_\mu \to \partial_\mu + ieA_\mu$. With only a point charge ze at the origin and a Dirac field of negative charge (-e), we have

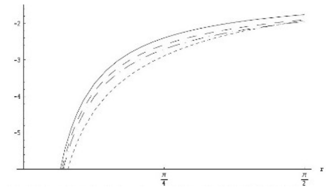
$$\left[\frac{\partial}{\partial r} \mp \left(\frac{\partial}{\partial\eta} - iz\alpha\frac{S'}{S}\right)\right]\Phi_{\pm} = -\left(\frac{\lambda}{S} \pm iMa\right)\Phi_{\mp} \quad \ldots(13)$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant and the minus sign is to represent attractive potential. The massless case M = 0 is easily solved with time dependence $e^{-i\omega\eta}$ to give the radial part as

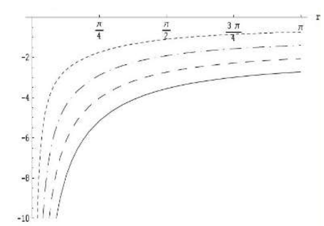
$$R_{\pm}(r) \sim (1 - i\cot r)^{-\left(b-\frac{1}{2}\right)}(1 + i\cot r)^{-\left(c-\frac{1}{2}\right)}P_n^{(-b,\ -c)}(i\cot r) \quad \ldots(14)$$

Where $\omega = n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \alpha^2 z^2}$, $b = \omega - i\alpha z \pm \frac{1}{2}$, and $c = \omega + i\alpha z \mp \frac{1}{2}$.

Plot of $|R_+|^2 + |R_-|^2$ representing the relative probability, for some typical values of az, l and n are shown in Fig. (1).



(a)



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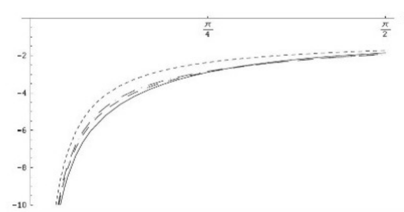


(c)

Figure 2: ***The modified vector potentials*** A_η ***with various parameters varied. (a) k = 0, a = 1, αz = 1 with l varied. Dotted for*** $l = 1/2$, ***dot-dashed for*** $l = 3/2$, ***dashed for*** $l = 5/2$ ***and solid for*** $l = 15/2$. ***(b) k = 0,*** $l = 1/2$ ***with αz varied. Dotted for αz = 0.5, dot-dashed for αz = 1, dashed for αz = 1.5 and solid for αz = 2. (c) k = 1,*** $l = 1/2$, $\alpha z = 1$ ***with a varied. Dotted for*** $a = 0.1$, ***dot-dashed for*** $a = 0.8$, ***dashed for a=1 and solid for*** $a = 1.5$.

For the massive case, although the equations can be decoupled for φ_+ and φ_-, we are unable to separate the variables r and η. We can get an insight into the behaviour by substituting $\frac{\lambda}{S} \pm iMa = \chi e^{\pm i\xi}$, where $\chi^2 = \frac{\lambda^2}{S^2} + M^2a^2$ and $\tan\xi = \frac{MaS}{\lambda}$, and $\psi_\pm = e^{\mp\frac{i\xi}{2}}\varphi$ to give

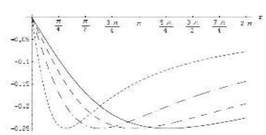
$$\left[\left(\frac{\partial}{\partial r} - \frac{i}{2}\frac{\partial\xi}{\partial\eta}\right) \mp \left(\frac{\partial}{\partial\eta} - iz\alpha\frac{S'}{S} - \frac{i}{2}\frac{\partial\xi}{\partial r}\right)\right]\psi_\pm = -\chi\psi_\mp \quad \ldots(15)$$

We can think of derivatives of ξ as adding on to the vector potentials as gravitational counter parts. In particular, the attractive modified potentials are

$$A_r = -\frac{1}{2}\frac{\partial\xi}{\partial\eta} = -\frac{1}{2}\sin\xi\cos\xi\,\frac{1}{a}\frac{\partial a}{\partial\eta} \quad \ldots(16)$$

$$A_\eta = -\alpha z\frac{S'}{S} - \frac{1}{2}\frac{\partial\xi}{\partial r} - \alpha z\frac{S'}{S}r - \frac{1}{2}\sin\xi\cos\xi\,\frac{S'}{S} \quad \ldots(17)$$

These potentials are plotted in Fig. (2) and (3).



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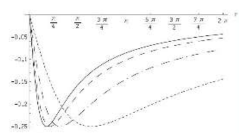

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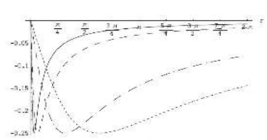

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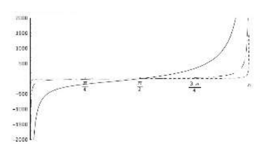


(d)

Figure 3: ***The modified vector potentials A_r with various parameters varied. (a) k = 0, a = 1, with l varied. Dotted for $l = 1/2$, dot-dashed for $l = 3/2$, dashed for $l = 5/2$ and solid for $l = 7/2$. (b) k = 0, $l = 1/2$ with a varied. Dotted for $a = 0.5$, dot-dashed for $a = 1$, dashed for a=1.5 and solid for $a = 1.8$. (c) k = 0, $l = 1/2$, $a = 1$ with m varied. Dotted for $m = 0.5$, dot-dashed for $m = 1$, dashed for m=5 and solid for $m = 10$. (d) k = 1, $l = 1/2$, $a = 1$ with αz varied. Dotted for αz = 1, dot-dashed for αz = 10 and solid for αz = 100.***

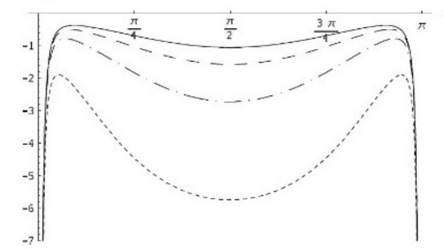
The force field due to these potentials is

$$F_{\eta r} = \frac{\partial A_r}{\partial\eta} - \frac{\partial A_\eta}{\partial r}$$

$$= -\frac{\alpha z}{S^2} - \frac{1}{2}\sin\xi\cos\xi\left[\frac{a''}{a} + K - 2\sin^2\xi\left(\left(\frac{a'}{a}\right)^2 + K - \frac{1}{S^2}\right)\right]$$

$$= -\frac{\alpha z}{S^2} - \frac{1}{2}\sin\xi\cos\xi\left[\frac{4\pi G}{3}a^2(\rho - 3P) - 2\sin^2\xi\left(\frac{8\pi G}{3}a^2\rho - \frac{1}{S^2}\right)\right]$$

This force field is also shown in Fig. (4) at different phases of expansion and various conditions.



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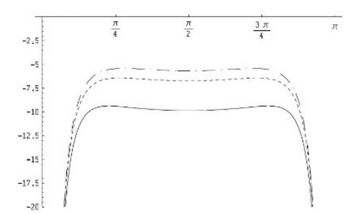

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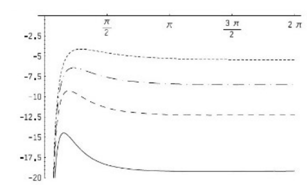

(c)

Figure 4: ***The force field $F_{\eta r}$ plotted against r. (a) matter-dominated closed (K=1) universe, $a = 1$, $\alpha z = .01$. Dotted for $l = 1/2$, dot-dashed for $l = 3/2$, dashed for $l = 5/2$ and solid for $l = 7/2$, (b) closed (K=1) universe, $l = 1/2$, $a = 1$, $\alpha z = 1$. Dotted for matter-dominated, dot-dashed for radiation- dominated, and solid for vacuum dominated, (c) matter- dominated, and solid for vacuum dominated, (c) matter-dominated flat (K=0) universe, $l = 1/2$, $\alpha z = 1$. Dotted for $a = 0.8$, dot-dashed for $a = 1$, dashed for $a =$ 1.2 and solid for $a = 1.5$.***

CONCLUSION AND DISCUSSION

In this work we have tried to investigate the behaviour of the Dirac particle in the FLRW spacetime endowed with Maxwell field. To do this, we first derived a representation for the Maxwellian vector potential that is analogous to the Lorentz and Coulomb gauges. Lorentz like gauge is quite evident, just requiring the vanishing of the four divergence of $\frac{A^\mu}{(aS)^2}$. The solution for the potentials become immediate in terms of the appropriate combinations of boost and spin weighted functions.

We also identify a Coulomb like gauge in which the potential reduces to $1/r$ in flat space, and solve for the appropriate Green function.

Then we use this condition to write the Dirac equation for an electron in the Coulomb like field of a point positive charge. We make the minimal substitution of the potential in the covariant derivative. The massless case can be completely solved, and the energy levels exhibit a quantized structure [Eq. 14]. But, we are not able to separate the radial and temporal parts of the equations for massive electron. Nonetheless, we are able to reduce the equations to where the scale factor and curvature of the FLRW spacetime appear as a gravitational counterpart, adding on to the Coulomb potential. Fig. 3 shows that this modified potential always develops a minimum, allowing bound states. Further work is necessary to determine whether this consequence will lead to a re-evaluation of the recombination era of the Universe.

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