A STUDY ON FIXED POINT THEORY IN M-METRIC SPACE

Prakash Muni Bajracharya and Nabaraj Adhikari

Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal

Abstract: In 2014, Asadi et al.[1] introduced the notion of an $M-$ metric space which is the generalization of a partial metric space and establish Banach and Kannan fixed point theorems in $M-$ metric space. In this paper, we give a brief survey regarding the fixed point theorem for Chatterjea contraction mapping in the framework of $M-$ metric space. We also give some examples which support the partial answers to the question posed by Asadi et al. concerning a fixed point for Chatterjea contraction mapping.

Keywords: Contraction principle; Partial metric spaces; M-metric spaces; Fixed Point

INTRODUCTION

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be consider as a core subject of non-linear analysis. In last 60 years, fixed point theory has been flourishing area of research for many mathematicians. The theory of fixed point was originated at the end of 19^{th} century to establish the existence and uniqueness of solutions particularly to differential equations using the successive approximations. This method is associated with many celebrated mathematicians, like Cauchy, Fredholm, Liouville, Lipchitz, Peano and Picard. Banach is credited as the starting point to metric fixed point theory. But the theory didn't gain enough impetus till Felix Browuer's major contribution to the development of the non-linear functional analysis as an active and vital branch of mathematics. In 1912, Brouwer's[3] prove the following fixed point theorem, which is called Brouwer's fixed point theorem.

Theorem 1.[3] *Every continuous mapping from unit ball in $\mathbb{R}^n$ into itself has a fixed point.*

Some authors, Schauder[10], Tychonoff[12], Kakutani[11] and many others have improved and generalized this theorem in several ways. In fact, in 1930 Schauder[13] prove fixed point theorem which is an extension of Brouwer's fixed point theorem to topological vector spaces, which states that

Theorem 2. (Schauder fixed point theorem)[10] *Let C be a non-empty compact convex subset of a normed linear space X. Then every continuous mapping from C into itself has a fixed point.*

Schauder fixed point theorem was generalized to locally convex topological vector space by Tychonoff[12], and this generalization is known as Schauder-Tychonoff theorem.

Theorem 3. (Schauder-Tychonoff fixed point theorem)[12] *Let T be compact and continuous mapping from a normed linear space X into itself and $T(X)$ is bounded. Then T has a fixed point.*

Now, we introduce various results on metric fixed point theory and its applications.

In 1890, Picard[9] prove the following theorem to show the existence of solutions for non-linear equations.

Theorem 4. (Picard Convergence Theorem)[9] *Let $T : [a,b] \to \mathbb{R}$ be a continuous function and $T : (a,b) \to \mathbb{R}$ be differentiable. If there exist $L < 1$ such that,*

$$|T'(x)| \leq L \tag{1}$$

for all $x \in (a,b)$, then the sequence (x_n) in (a,b) defined by,

$$x_{n+1} = Tx_n \tag{2}$$

for all non-negative integer n converges to a solution of the equation $Tx = x$.

This iterative sequence (x_n) defined by (2) is called Picard iterative sequence.

In 1922, Banach[2] proved a theorem which is well known as Banach's fixed point theorem to establish the existence of solution for integral equations.

Theorem 5. (Banach's fixed point theorem) Let (X,d) be a complete metric space and $T : X \to X$ be a contractive mapping, that is, there exist $\alpha \in [0,1)$ such that,

$$d(Tx,Ty) \leq \alpha d(x,y) \tag{3}$$

for all $x,y \in X$. Then T has a unique fixed point $x \in X$. Furthermore, for each $x_0 \in X$, the sequence (x_n) defined by,

$$x_{n+1} = Tx_n$$

for all non-negative integer n converges to the fixed point x of T.

It can prove that Picard convergence theorem and Banach's fixed point theorem are equivalent.

Further, since Banach's fixed point theorem, because of its simplicity, usefulness, and applications, it has become a very popular tool in solving the existence problems in many branches of mathematical analysis. Recently, many authors have improved, extended, and generalized Banach's fixed point theorem in the following ways.

First, how to generalize Banach's contraction? Second, how to extend Banach's fixed point theorem in metric spaces to the large class of various spaces? Third, how to generalize the Picard iterative sequence? Fourth, how to apply Banach's fixed point theorem to applied mathematics and others? Fifth, does the converse of Banach's fixed point theorem hold?

Author for Correspondence : Nabaraj Adhikari, Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.
Email: topology88@gmail.com

Generalization of Contractive Mapping

Example 1. Let $X = \{x \in \mathbb{R} : x \geq 1\}$ with metric

$$d(x,y) = |x - y|, \qquad \forall x,y \in X,$$

and let $T : X \to X$ be given by

$$T(x) = x + \frac{1}{x}.$$

Then,

$$d(Tx,Ty) = \frac{xy-1}{xy}|x-y| < |x-y| = d(x,y)$$

On the other hand, there does not exist $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y)\ \forall x,y \in X,$$

and we can see that T has no fixed points in X, since $Tx = x + \frac{1}{x} \neq x$. This shows that if we replace the assumption of the theorem that T be a contraction mapping by the weaker hypothesis

$$d(Tx,Ty) < d(x,y),\ \forall x,y \in X,$$

then T need not have a fixed point.

Despite such example, in 1962, Edelstein [5] proved following fixed point theorem.

Theorem 6. (Edelstein's fixed point theorem) Let (X,d) be a compact metric space and $T : X \to X$ be mapping satisfying the following contraction condition (weaker than condition (3))

$$d(Tx,Ty) < d(x,y) \tag{4}$$

for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

From Banach's contraction condition (BC), it follows that the mapping T is continuous. Further, we use the continuity of the mapping T to prove Banach's fixed point theorem. Thus, it is natural to consider the following question:

Do there exist some contractive conditions which do not force the mapping T to be continuous? In 1968, Kannan[6] gave the positive answers for this question by proving the following fixed point theorem for contractive conditions in complete metric space, which is called Kannan contraction condition.

Theorem 7. (Kannan's fixed point theorem) Let (X,d) be a complete metric space and $T : X \to X$ be mapping such that there exist a number $k \in [0, \frac{1}{2})$ such that,

$$d(Tx,Ty) \leq k[d(Tx,x) + d(Ty,y)] \tag{5}$$

for all $x,y \in X$. Then T has a unique fixed point in X. Furthermore, for each $x_0 \in X$, the sequence (x_n) is defined by,

$$x_{n+1} = Tx_n$$

for all non-negative integer n converges to the fixed point x of T.

It can be shown that Kannan contraction mapping may not be continuous. This is the big difference between Banach's and Kannan's contraction mapping. Now, we give one example that a mapping T is Kannan's contraction but not continuous.

Example 2. Let $X = \mathbb{R}$ be a usual metric space and $T : X \to X$ be a mapping defined by,

$$Tx = \begin{cases} 0 & \text{if } x \in (-\infty, 2] \\ 1/2 & \text{if } x \in (2, \infty) \end{cases}$$

Then T is not continuous on $\mathbb{R}$, but it satisfies Kannan's contraction (KC) with $k = \frac{1}{5}$.

In 1972, Chatterjea[4] introduced the contractive condition called Chatterjea contractive condition (CHC) and prove the following fixed point theorem:

Theorem 8. (Chatterjea's fixed point theorem) Let (X,d) be a complete metric space and $T : X \to X$ be mapping such that there exist a number $h \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq h[d(x,Ty) + d(y,Tx)] \tag{6}$$

for all $x,y \in X$. Then T has a unique fixed point in X. Furthermore, for each $x_0 \in X$, the sequence (x_n) is defined by

$$x_{n+1} = Tx_n$$

for all non-negative integer n converges to the fixed point x of T.

It can be shown that Banach's contraction (BC), Kannan's contraction (KC) and Chatterjea contraction (CHC) are independent.

Consequences of Banach Fixed Point Theorem

In this section we are discussing about the extension of Banach's Kannan's and Chatterjea's fixed point theorem in generalized metric spaces for instance: few of them are cone metric spaces, partially ordered metric spaces, fuzzy metric spaces, complex valued metric spaces, probabilistic metric spaces, random normed spaces, ordered Banach spaces, bmetric spaces, 2-metric spaces, G-metric spaces, M-metric spaces, S-metric spaces, and other spaces. More precisely, we are intended to mention *partial metric space* and *M-metric space* and study fixed point theory and its applications in these metric spaces.

Fixed Point Theorems in Partial Metric Space

Especially, in 1994, Matthews[8] extended the concept of a metric to *partial metric* and introduced the notion of partial metric space. Indeed, the motivation for introducing the concept of a partial metric was to obtain appropriate mathematical model in the theory of computation. Also, he obtained many results in partial metric spaces. In particular, he investigated the improvement of Banach's contraction principle in the sense of partial metric spaces. Afterward, many mathematicians have studied the existence and uniqueness of a fixed point for nonlinear mapping satisfying various contractive conditions in the setting of partial metric spaces.

Definition.[8] A partial metric on a non-empty set X is a function $p : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$:

(p1.) $p(x,x) = p(y,y) = p(x,y)$ if and only if

$x = y$ (equality)

(p2.) $0 \leq p(x,x) \leq p(x,y)$ (small self-distances)

(p3.) $p(x,y) = p(y,x)$ (symmetry)

(p4.) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (triangularity)

Then p is said to partial metric or a distance function on X, and a pair (X,p) is called partial metric space.

It is easy to see that a metric d is also a partial metric p, but the converse may not be true.

Example 3. Let $X = [0,\infty)$ and $p : X \times X \to [0,\infty)$ be defined by

$$p(x,y) = max\{x,y\}$$

for all $x,y \in X$. Then p is a partial metric on X but it is not metric on X. Indeed, $p(1,1) = 1 \neq 0$

Example 4. Let $X = \{[a,b] : a,b \in \mathbb{R}\}$ and p: $X \times X \to [0,\infty)$ be defined by

$$p([a,b],[c,d]) = max\{b,d\} - min\{a,c\}$$

for all $[a,b],[c,d] \in X$. Then p is a partial metric on X but it is not metric on X. Since, $p([1,2],[1,2]) = 1 \neq 0$

Matthews[8] obtained the following Banach's, Kannan's and Chatterjea's fixed point theorem on a complete partial metric space.

Theorem 9. (Banach's fixed point theorem) Let T be a mapping on complete partial metric space (X,p) into itself such that there is a real number k with $k \in [0,1)$, satisfying for all $x,y \in X$:

$$p(Tx,Ty) \leq kp(x,y). \tag{7}$$

Then T has a unique fixed point in X.

Theorem 10. (Kannan's fixed point theorem) Let (X,d) be a complete partial metric space and $T : X \to X$ be mapping such that there exist a number $k \in [0, \frac{1}{2})$ such that,

$$p(Tx,Ty) \leq k[p(Tx,x) + p(Ty,y)] \tag{8}$$

for all $x,y \in X$. Then T has a unique fixed point in X.

Theorem 11. (Chatterjea's fixed point theorem) Let (X,d) be a complete partial metric space and $T : X \to X$ be mapping such that there exist a number $h \in [0, \frac{1}{2})$ such that $p(Tx,Ty) \leq h[p(x,Ty) + p(y,Tx)]$ (9) for all $x,y \in X$. Then T has a unique fixed point in X.

Hence Banach's, Kannan's and Chatterjea's fixed points are also true in complete partial metric space.

Fixed Point Theorems in M-Metric Spaces

Based on the result of Maththews, in 2014, M. Asadi et al.[1] introduced the concept of an M-metric space which is a generalization of a partial metric space. They studied topological properties in such spaces and established some fixed point results in M-metric spaces, which are generalizations of Banach's and Kannan's fixed point theorems in the framework of partial metric space as follows:

Theorem 12.[1]*. Let (X,m) be a complete $M-$ metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$\exists k \in [0,1)$ such that $m(Tx,Ty) \leq km(x,y)$ for all $x,y \in X$. Then T has a unique fixed in X. Furthermore, for each $x_0 \in X$, the sequence (x_n) defined by

$$x_{n+1} = Tx_n$$

for all non-negative integer n converges to the fixed point x of T.

Theorem 13.[1] *Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition :*

$$\exists k \in [0, \frac{1}{2}) : m(Tx,Ty) \leq k[m(x,Tx) + m(y,Ty)]$$

for all $x,y \in X$. Then T has an unique fixed point in X.

Moreover, for each $x_0 \in X$, the sequence (x_n) defined by

$$x_{n+1} = Tx_n$$

for all non-negative integer n converges to the fixed point x of T.

Existence and uniqueness of fixed point for Chatterjea contraction mapping in the framework of M-metric space was unsolved and uncertain at their study. As a consequence, they posed the following open problem:

Problem 1.[1] Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition

$$\exists k \in [0, \frac{1}{2}) : m(Tx,Ty) \leq k[m(x,Ty) + m(Tx,y)]$$

for all $x,y \in X$. Does T have a unique fixed point?

In this work, we study a partial answers to the Problem (1). Furthermore, we give some illustrative examples which support partial answers.

Preliminaries:

The following definitions, notations and lemmas are needed in the sequel:

For a non-empty set X and a function $m : X \times X \to [0,\infty)$.

The following notation is useful in the sequel:

(1.) $m_{xy} = \min\{m(x,x),m(y,y)\}$

(2.) $M_{xy} = \max\{m(x,x),m(y,y)\}$

Definition. [1] Let X be a non-empty set. A function $m : X \times X \to [0,\infty)$ is called an m-metric if the following conditions are satisfied for all $x,y,z \in X$:

- *(MM1)* $m(x,x) = m(y,y) = m(x,y)$ if and only if $x = y$
- *(MM2)* $m_{xy} \leq m(x,y)$
- *(MM3)* $m(x,y) = m(y,x)$
- *(MM4)* $[m(x,y) - m_{xy}] \leq [m(x,z) - m_{xz}] + [m(z,y) - m_{zy}]$

A set X with a metric m defined on it is called a M-metric space. It is denoted by (X,m).

Remark. According to the definitions of a $p-$ metric and an m-metric, we have the following consequences

1. The condition $(p1)$ in definition of partial metric is same to the condition $(MM1)$ in the definition of m-metric.

2. The condition ($p2$) for $p(x,x)$ is expressed just $p(y,y) = 0$ (we may have $p(y,y)$ 6= 0) and so the condition ($p2$) is replaced by $\min\{p(x,x),p(y,y)\} \leq p(x,y)$, that is, the condition ($MM2$).
3. The condition ($p3$) is same to the condition ($MM3$).
4. Also, they improve the condition ($p4$) to the form of ($MM4$).

Thus, every $p-$ metric is an $M-$ metric. But converse may not be true as shown in the following examples:

Remark. For all $x,y \in X$

1. $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$
2. $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$

Example 5. Let $X = [0,\infty)$ and a function $m : X \times X \to [0,\infty)$ be defined by

$$m(x,y) = \frac{x+y}{2}$$

for all $x,y \in X$. Then m is $M-$metric on X but it is not a partial metric . For example, $m(4,4) = 4 > 3 = m(4,2)$

Example 6. Let $X = \{1,2,3\}$. Define $m(1,1) = 1$, $m(2,2) = 9$, $m(3,3) = 5$, $m(1,2) = m(2,1) = 10$, $m(1,3) = m(3,1) = 7$, $m(3,2) = m(2,3) = 7$. Then m is M- metric space but it is not a partial metric space because it does not satisfy the triangle inequality:

$m(1,2) \geq m(1,3) + m(3,2) - m(3,3)$.

Thus, we obtain the following relation:

$$\text{metric} \Rightarrow \text{ partial metric} \Rightarrow M- \text{ metric.}$$

Lemma 14. If m is an $M-$ metric on a non-empty set X, then the function $d_m : X \times X \to [0,\infty)$

defined by

$$d_m(x,y) = m(x,y) - 2m_{xy} + M_{xy}$$

is metric on X.

Now, we give the concepts of a convergent sequence, Cauchy sequence and the completeness in M-metric spaces.

Definition.[1] Let (X,m) be a M-metric space.

(1) A sequence (x_n) in a M-metric space (X,m) *converges* to a point $x \in X$ if and only if

$$\lim_{n\to\infty}[m(x_n,x) - m_{x_nx}] = 0.$$

(2) A sequence (x_n) in a M-metric space (X,m) is called an m- Cauchy sequence if $\lim_{m,n\to\infty}[m(x_n,x_m) - m_{x_nx_m}]$ and $\lim_{m,n\to\infty}(M(x_n,x_m) - m_{x_nx_m})$ exist and are finite.

(3) An $M-$ metric space (X,m) is said to be *complete* if every m- Cauchy sequence (x_n) in X converges to a point $x \in X$ such that $\lim_{n\to\infty}[m(x_n,x) - m_{x_nx}] = 0$, and $\lim_{n\to\infty}[M(x_n,x) - m_{x_nx}] = 0$

Lemma 15.[1] Let (X,m) be an M- metric space. Then

(1) *(x_n) is a Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,d_m).*

(2) *(X,m) is complete if and only if the metric space (X,d_m) is complete. Furthermore, for each (x_n) in X and $x \in X$*

$$\lim_{n\to\infty} d_m(x_n,x) = 0$$

$$\iff \lim_{n\to\infty}[m(x_n,x) - m_{x_nx}] = 0$$

$$and \quad \lim_{n\to\infty}[M(x_n,x) - m_{x_nx})] = 0 \tag{10}$$

Example 7. Let $X = [0,\infty)$ and a function $m : X \times X \to [0,\infty)$ be defined by

$$m(x,y) = \frac{x+y}{2}$$

for all $x,y \in X$. Then (X,m) is a complete $M-$metric space.

Proof. For $x,y \in X$ we have

$$d_m(x,y) = m(x,y) - 2m_{xy} + M_{xy}$$
$$= \frac{x+y}{2} - 2\min\{x,y\} + \max\{x,y\} \tag{11}$$

We consider the following two cases:

Case 1: Let $x \geq y$. From equation (11), we get

$$d_m(x,y) = \frac{x+y}{2} - 2y + x = \frac{3}{2}|x-y|$$

Case 2: Let $x < y$. From equation (11), we get

$$d_m(x,y) = \frac{x+y}{2} - 2x + y = \frac{3}{2}|x-y|$$

In each case, we obtain $d_m(x,y) = \frac{3}{2}|x-y| \ \forall x,y \in X$. Since X with usual metric is a complete metric space, (X,d_m) is also a complete metric space. From Lemma (15) (X,d_m) is a complete $M-$metric space.

Lemma 16.[7] Let (x_n) be a sequence in an $M-$metric space (X,m) such that $\exists r \in [0,1)$ and $m(x_{n+1},x_n) \leq rm(x_n,x_{n-1})$,$\forall n \in$ N.

Then

(A1.) $\lim_{n\to\infty} m(x_n,x_{n-1}) = 0$

(A2.) $\lim_{n\to\infty} m(x_n,x_n) = 0$

(A3.) $\lim_{m,n\to\infty} m_{x_mx_n} = 0$

(A4.) (x_n) *is an $m-$Cauchy sequence.*

Theorem 17.[7] Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following conditions :

$$\exists k \in [0,\frac{1}{2}) : m(Tx,Ty) \leq k[m(x,Ty)+m(Tx,y)] \tag{12}$$

for all $x,y \in X$. If there is $x_0 \in X$ such that

$$m(T^nx_0,T^nx_0) \leq m(T^{n-1}x_0,T^{n-1}x_0) \tag{13}$$

for all $n \in \mathbb{N}$, then T has a unique fixed point. Moreover, if the Picard sequence (x_n) in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ such that x_0 is an initial point in the condition (12) then (x_n) converges to a fixed point of T.

Proof. Starting from $x_0 \in X$ in the hypothesis, we will construct the sequence (x_n) in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. From the given conditions (12), (13) and (MM4), we get

$$\begin{aligned} m(x_{n+1},x_n) &= m(Tx_n,Tx_{n-1}) \\ &\le k[m(x_n,x_n) + m(x_{n+1},x_{n-1})] \\ &\le k[m(x_n,x_n) + m(x_{n+1},x_n) \\ &\quad - m_{x_{n+1}x_n} + m(x_n,x_{n-1}) \\ &\quad - m_{x_nx_{n-1}} + m_{x_{n+1}x_{n-1}}] \\ &= k[m(x_n,x_n) + m(x_{n+1},x_n) \\ &\quad - m(x_{n+1},x_{n+1}) + m(x_n,x_{n-1}) \\ &\quad - m(x_n,x_n) + m(x_{n+1},x_{n+1})] \\ &= k[m(x_{n+1},x_n) + m(x_n,x_{n-1})] \end{aligned}$$

for all $n \in \mathbb{N}$.

This implies that

$$m(x_{n+1},x_n) \le rm(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$, where $0 \le r = \frac{k}{k-1} < 1$. Hence by using Lemma (16) we get (A_1),(A_2),(A_3) and (A_4) in Lemma (16) holds. It follows from (A_4) that (x_n) is an $M-$Cauchy sequence in X. From the completeness of X, we get $x_n \to x$ as $n \to \infty$ for some $x \in X$. So

$$m(x_n,x) - m_{x_nx} \to 0 \quad as \quad n \to \infty \tag{14}$$

and

$$M_{x_nx} - m_{x_nx} \to 0 \quad as \quad n \to \infty. \tag{15}$$

From (A_2), we get $m(x_n,x_n) \to 0$ as $n \to \infty$ and so

$$m_{x_nx} = \min\{m(x_n,x_n),m(x,x)\} \to 0 \text{ as } n \to \infty \tag{16}$$

and

$$m_{x_nTx} = \min\{m(x_n,x_n),m(Tx,Tx)\} \to 0 \text{ as } n \to \infty \tag{17}$$

From (14), (15) and (16), we obtain

$$m(x_n,x) \to 0 \text{ as } n \to \infty \tag{18}$$

and

$$M_{x_nx} \to 0 \text{ as } n \to \infty. \tag{19}$$

By Remark, we have

$$M_{x_nx} + m_{x_nx} = m(x_n,x_n) + m(x,x) \forall n \in \mathbb{N}. \tag{20}$$

Taking limit as $n \to \infty$ in the above equation and using (16), (19) and (A_2), we have

$$m(x,x) = 0 \tag{21}$$

This implies that

$$m_{xTx} = \min\{m(x,x),m(Tx,Tx)\} = 0. \tag{22}$$

Next, we will show that $m(x,Tx) = 0$. From (MM4) we get

$$\begin{aligned} m(x,Tx) = m(x,Tx) - m_{xTx} &\le m(x,x_n) \\ &-m_{xx_n} + m(x_n,Tx) - m_{x_nTx} \end{aligned} \tag{23}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \to \infty$ in (23) and using (14), (15), (17) and (22), we get

$$\begin{aligned} m(x,Tx) & \\ &\le \lim_{n\to\infty} \sup[m(x,x_n) - m_{xx_n} + m(x_n,Tx) - m_{x_nTx}] \\ &\le \lim_{n\to\infty} \sup[m(x,x_n) - m_{xx_n} + m(x_n,Tx)] \\ &\le \lim_{n\to\infty} \sup[m(x,x_n) - m_{xx_n}] + \lim_{n\to\infty} \sup m(x_n,Tx) \\ &= \lim_{n\to\infty} \sup[m(x_n,Tx)] \\ &\le \lim_{n\to\infty} \sup[k(m(x_{n-1},Tx) + m(Tx_{n-1},x))] \\ &\le k[\lim_{n\to\infty} \sup m(x_{n-1},Tx) + \lim_{n\to\infty} \sup m(x_n,x)] \\ &= k[\lim_{n\to\infty} \sup m(x_{n-1},Tx)] \\ &\le k\left[\lim_{n\to\infty} \sup m(x_{n-1},x) - m_{x_{n-1}x}\right] \\ &+ k\left[m(x,Tx) - m_{xTx} + m_{x_{n-1}Tx}\right] \\ &\le km(x,Tx) \end{aligned}$$

This implies that

$$m(x,Tx) = 0. \tag{24}$$

By the contractive condition (12), we have

$$m(Tx,Tx) \le 2km(x,Tx) = 0 \tag{25}$$

and hence

$$m(Tx,Tx) = 0. \tag{26}$$

From (16), (24) and (25), we obtain

$$m(x,x) = m(Tx,Tx) = m(x,Tx).$$

Using the property (MM1), we get $x = Tx$.

Uniqueness:

Let y be an another fixed point of T. From condition (12), we get

$$\begin{aligned} m(x,y) &= m(Tx,Ty) \\ &\le k(m(x,Ty) + m(y,Tx)) \\ &= k(m(x,y) + m(y,x)) \\ &\le 2km(x,y) \\ &< m(x,y) \end{aligned}$$

which is contradiction. Hence T has a unique fixed point.

Example 8. Let $X = [0,\infty)$ and a function $m : X \times X \to [0,\infty)$ be defined by

$$m(x,y) = \frac{x+y}{2}$$

for all $x,y \in X$. Then (X,m) is complete $M-$metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} 0 & \text{if } 0 \le x < 3 \\ \frac{x}{1+x} & \text{if } x \ge 3. \end{cases}$$

Then T has unique fixed point in X.

Proof. We will show that T satisfy the general contractive condition (12) of Theorem (17) with $k = \frac{1}{4}$. Let $x,y \in X$.

Then there are three possible cases:

Case I: If $x,y \in [0,3)$ then $Tx = 0 = Ty$ so the condition $m(Tx,Ty) \leq k[m(x,Ty) + m(Tx,y)]$ is true for all k.

Case II: If $x,y \in [3,\infty)$, we get

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{x}{1+x} + \frac{y}{1+y}\right) \\ &\leq \frac{1}{2}(\frac{x}{4} + \frac{y}{4}) \\ &\leq \frac{1}{4}(\frac{x}{2} + \frac{y}{2(1+y)} + \frac{y}{2} + \frac{x}{2(1+x)}) \\ &= k[m(x,Ty) + m(y,Tx)] \end{aligned}$$

with $k = \frac{1}{4}$.

Case III: Let $(x,y) \in [3,\infty) \times [0,3) \bigcup [0,3) \times [3,\infty)$. Without loss of generality, we may assume that $x \in [0,3)$ and $y \in [3,\infty)$, we get

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{y}{1+y}\right) \\ &\leq \frac{1}{2}(\frac{y}{4}) \\ &\leq \frac{1}{4}(0 + \frac{y}{2(1+y)} + \frac{y}{2} + 0) \\ &= k[m(x,Ty) + m(y,Tx)] \end{aligned}$$

Then T satisfies the condition (12) of Theorem (17) for all $x,y \in X$ with $k = \frac{1}{4}$. Also, T satisfies the condition (13) for all $x_0 \in X$. Thus, all condition of Theorem (17) are satisfied and so there exist a unique fixed point of T. In this case 0 is a unique fixed point of T.

Example 9. Let $X = [0,\infty)$ and a function $m : X \times X \to [0,\infty)$ be defined by

$$m(x,y) = \frac{x+y}{2}$$

for all $x,y \in X$. Then from example (7) (X, m) is complete M−metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} x^2 & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{4} & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Then T has unique fixed point in X.

Proof. We will show that T satisfied the general contractive condition of Theorem (17) with $k = \frac{1}{3}$. Let $x,y \in X$. Then there are three possible cases:

Case I: If $x, y \in [0, \frac{1}{2})$, then we have

$$\begin{aligned} m(Tx,Ty) &= \frac{x^2}{2} + \frac{y^2}{2} \\ &\leq \frac{x}{6} + \frac{y^2}{6} + \frac{y}{6} + \frac{x^2}{6} \\ &= \frac{1}{3}(\frac{x}{2} + \frac{y^2}{2} + \frac{y}{2} + \frac{x^2}{2}) \\ &= \frac{1}{3}[m(x,Ty) + m(y,Tx)]. \end{aligned}$$

Case II: If $x, y \in [\frac{1}{2}, \infty)$ then we have

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{8} + \frac{1}{8} \\ &\leq \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + \frac{1}{24} \\ &= \frac{1}{3}(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{1}{8}) \\ &= \frac{1}{3}[m(x,Ty) + m(y,Tx)] \end{aligned}$$

Case III: Let $(x, y) \in [0, \frac{1}{2}) \times [\frac{1}{2}, \infty) \bigcup [\frac{1}{2}, \infty) \times [0, \frac{1}{2})$.

Without loss of generality, we may assume that $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, \infty)$. We have $\frac{x^2}{2} \leq \frac{x}{6} + \frac{x^2}{6}$ and $\frac{1}{8} \leq \frac{1}{24} + \frac{y}{6}$.

Then we get

$$\begin{aligned} m(Tx,Ty) &= \frac{x^2}{2} + \frac{1}{8} \\ &\leq \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + + \frac{x^2}{2} \\ &= \frac{1}{3}(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{x^2}{2}) \\ &= \frac{1}{3}[m(x,Ty) + m(y,Tx)]. \end{aligned}$$

Then T satisfies the condition (12) of Theorem (17) for all $x,y \in X$ with $k = \frac{1}{3}$. Also, T satisfies the condition (13) of Theorem (17) for all $x_0 \in X$. Thus, all conditions of Theorem (17) are satisfied and so there exist a unique fixed point of T. In this case 0 is a unique fixed point of T.

CONCLUSION AND OPEN PROBLEM

In this paper, based on the fixed point results of Aasadi et al.[1], we have studied the fixed point theorem for Banach, Kannan and Chatterjea contraction mapping in M-metric spaces. We give two examples to illustrate the validity of Theorem (17).

Based on[9], we have the the following open problem:

Problem 2. Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition :

$$\exists k \in [0, \frac{1}{2}) : m(Tx,Ty) \leq k[m(x,Ty) + m(Tx,y)] \quad (27)$$

for all $x,y \in X$. If there is $x_0 \in X$ such that

$$m(T^n x_0, T^n x_0) \leq m(T^{n-1} x_0, T^{n-1} x_0) \quad (28)$$

for all $n \in \mathbb{N}$, then T has a unique fixed point.

Can this problem be solved without the condition (28)?

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