Some Fixed Point Results for Two Pairs of Mappings on Integral and Rational Settings

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ABSTRACT

In 2000, P. Hitzler and A.K. Seda (Hitzeler & Seda 2000) obtained a very important generalization of topology which they named as dislocated topology. The corresponding generalized notion of metric obtained from dislocated topology was named as dislocated metric.The fixed point theorem for a single map satisfying contractive condition of integral type with a summable Lebesgue integrable mapping in complete metric space was first time estabished by Branciari (Branciari 2002) in the year 2002. B. E. Rhoades (Rhoades 2003) further extended the theorem of Branciari (Branciari 2002) with a most general contractive condition. Extensions and generalizations for rational and integral type mapping in various spaces can be seen in the literature of fixed point theory. This article establishes some common fixed point results satisfying integral and rational type contractive conditions with common limit range property for two pairs of maps in dislocated metric space. We have established common fixed point result in dislocated metric space with compatible and reciprocal continuity of mappings.

Keywords: Common fixed point, Dislocated metric space, Weakly compatible maps, Cauchy sequence

1. INTRODUCTION

In 1886 A.D., H. Poincare first time introduced the notion of fixed point. In 1922, an important and remarkable result was presented by S. Banach (Banach 1922) for a contraction mapping in complete metric space which is famous as Banach Contraction Principle (BCP). After establishment of BCP, various generalizations by several authors are obtained in the literature of fixed point theory.Now, the theory of fixed point has become a most crucial and dynamic area of research in nonlinear analysis. A remarkable generalization of BCP was obtained by A. Meir and E. Keeler (Meir & Keeler 1969) with (ϵ-δ) notions.

The concept of compatible maps was initiated by G. Jungck (Jungck 1986). R. P. Pant (Pant 1999) introduced the concept of reciprocally continuous mappings in metric space. P. Hitzler and A. K. Seda (Hitzeler & Seda 2000) obtained a generalization of topology which they named as dislocated topology.The corresponding generalized notion of metric from dislocated topology is dislocated metric. However, The concept of dislocated metric space appeared in (Matthews 1986) by S. G. Matthews under metric domains.

Branciari (Branciari 2002) obtained a fixed point theorem for a single map satisfying contractive condition of integral type with a summable Lebesgue integrable mapping in complete metric space. B. E. Rhoades (Rhoades 2003) extended the theorem of Branciari (Branciari 2002) with a most general contractive condition. In 2005, F.M Zeyada, G.H. Hassan and M.A. Ahmed (Zeyada *et al.* 2005) introduced dislocated quasi metric space and established fixed point theorems. In 2008, C.T. Aage and J. N. Salunke (Aage & Salunke 2008) proved some results on fixed points in dislocated and dislocated quasimetric spaces. In 2010, A. Isufati (Isufati 2010) proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric space. After these theorems in the literature, several authors have generalized and extended the fixed point results in various spaces for different types of contractive conditions and mappings in dislocated metric space.

The purpose of this article is to establish some fixed point theorems using common limit range property for compatible maps, weakly compatible maps and reciprocally continuous maps having integral and rational type contractive conditions in dislocated metric space.

2. PRELIMINARIES

We start with the following definitions, lemmas and theorems.

Definition 1 (Hitzler & Seda 2000) Let X be a non empty set and let $d:X\times X\to[0,\infty)$ be a function satisfying the following conditions:

1. $d(x,y) = d(y,x)$
2. $d(x,y)= d(y,x) = 0$ implies $x = y$
3. $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called dislocated metric (or d-metric) on X and the pair (X, d) is called the dislocated metric space (or d-metric space).

Definition 2 (Meir & Keeler 1969) A self - mapping T of a metric space (X, d) is called a weakly uniformly strict contraction or simply an (ε-δ) contraction if for each ε>0 there exists δ>0 such that for all $x,y\in X$

$$\varepsilon\le d(x,y)<\varepsilon+\delta \Rightarrow d(Tx,Ty)<\varepsilon \qquad (1)$$

Theorem 1 (Meir & Keeler 1969) Let (X, d) be a complete metric space and $T:X\to X$ is weakly uniformly strict contraction then T has a unique common fixed point, say z and for any

$x \in X$, $\lim_{n\to\infty} T^n x = z$.

Theorem 2 (Branciari 2002) Let (X, d) be a complete metric space, $c \in (0,1)$, and let $f:X\to X$ be a mapping such that for each $x,y\in X$,

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{d(x,y)} \varphi(t)\,dt$$

where $\varphi:[0,+\infty)\to[0,+\infty]$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon >0, \int_0^{\varepsilon} \varphi(t)\,dt$; then f has a unique fixed point $a\in X$ such that for each $x\in X$, $\lim_{n\to+\infty} f^n x=a$.

Theorem 3 (Rhoades 2003) Let (X,d) be a complete metric space, $k\in(0,1)$, and let $f:X\to X$ be a mapping such that for each $x,y\in X$

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le k\int_0^{m(x,y)} \varphi(t)\,dt$$

where,

$$m(x,y)=max\left\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\right\}$$

and φ:[0,+∞)→[0,+∞] is a Lebesgue-integrable mapping which is summable (i.e., with finite integral), nonnegative, and such that

$$\int_0^{\varepsilon}\varphi(t)dt>0;\ for\ each\ \varepsilon>0$$

then f has a unique fixed point z ∈X, and for each x∈X, $\lim_n f^n x = z$.

Definition 3 (Jungck 1986) *Two mappings S and T from a metric space (X, d) into itself are called compatible if* $\lim_{n\to\infty} d(STx_n, TSx_n)=0$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Tx_n=x$ for some x ∈ X

Definition 4 *Let A and S be two self-mappings on a set X. If Ax = Sx for some x∈ X, then x is called the coincidence point of A and S.*

Definition 5 (Jungck & Rhoades 1986) *Let A and S be mappings from a metric space (X,d) into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is, Ax=Sx for some x ∈ X implies ASx=SAx.*

Definition 6 (Pant 1999) *Two self mappings A and S of a metric space (X, d) are called reciprocally continuous if* $lim_n ASx_n = At$ *and* $lim_n SAx_n = St$ *whenever* $\{x_n\}$ *is a sequence such that* $lim_n Ax_n = lim_n Sx_n = t$ *for some* $t \in X$.

If A and S both are continuous they are obviously reciprocally continuous but the converse is not true.

Definition 7 (Sintunavarat & Kummam 2011) *Let A and S be two self-mappings defined in a metric space (X, d). We say that the two mappings A and S satisfy common limit range property* (CLR_A) *if there exists a sequence* $\{x_n\}$ *in X such that*

$$lim_n Ax_n = lim_n Sx_n = Ax\ \ for\ some\ \ x\in X$$

Lemma 1 (Jachymski 1994) *Let (X, d) be a metric space. Let A,B,S,T:X→X be mappings satisfying the conditions*

$$A(X)\subset T(X)\ and\ B(X)\subset S(X) \qquad (2)$$

Assume further that given for each ε > 0 there exists δ > 0 such that for all $x,y \in X$

$$\varepsilon<M(x,y)<\varepsilon+\delta\Rightarrow d(Ax,By)\le\varepsilon \qquad (3)$$

and

$$d(Ax,By)\langle M(x,y)\quad whenever\ M(x,y)\rangle 0 \qquad (4)$$

where

$$M(x,y)=max\left\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{[d(Sx,By)+d(Ax,Ty)]}{2}\right\}$$

then for each $x_0 \in X$, the sequence $\{y_n\}$ in X defined by the rule

$$y_{2n}=Ax_{2n}=Tx_{2n+1}, y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}$$

is a Cauchy sequence.

3. MAIN RESULTS

We establish a common fixed point theorem satisfying integral and rational type contractive condition with common limit range (CLR) property for two pairs of weakly compatible maps in dislocated metric space.

Theorem 4 *Let (X,d) be a dislocated metric space. Let A,B,S,T:X→X satisfying the following conditions*

$$A(X)\subseteq S(X)\ and\ B(X)\subseteq T(X) \qquad (5)$$

$$\int_0^{d(Ax,By)}\phi(t)dt<\int_0^{M(x,y)}\phi(t)dt) \qquad (6)$$

where,

$$M(x,y)=Min\{\frac{d(Tx,Sy)d(Tx,Ax)}{d(Tx,By)},d(Tx,Sy),d(Tx,Ax),d(By,Sy)$$

$$d(Tx,By),d(Sy,Ax),\frac{d(By,Sy)d(Ax,Sy)}{d(Tx,By)}\}$$

$$d(Tx,By),d(Sy,Ax),\frac{d(By,Sy)d(Ax,Sy)}{d(Tx,By)}\} \qquad (7)$$

$\phi:\mathbb{R}^+\to\mathbb{R}^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon}\phi(t)dt>0\quad for\ each\ \epsilon>0 \qquad (8)$$

1. The pairs (A,T) or (B,S) satisfy (CLR)-property

2. The pairs (A,T) and (B,S) are weakly compatible

then the pairs (A, T) and (B, S) have coincidence point and the four maps A,B,S and T have an unique common fixed point.

Proof: Assume that the pair (A,T) satisfy (CLR_A) property, so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = Ax \quad (9)$$

for some $x \in X$. Since $A(X) \subseteq S(X)$, so there exists a sequence $\{y_n\} \in X$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sy_n = Ax$. We show that

$$\lim_{n\to\infty} Bx_n = Ax \quad (10)$$

From the relation (6) we have

$$\int_0^{d(Ax_n,By_n)} \phi(t)\,dt < \int_0^{M(x_n,y_n)} \phi(t)\,dt, \quad (11)$$

where,

$$M(x_n,y_n) = Min\{\frac{d(Tx_n,Sy_n)\,d(Tx_n,Ax_n)}{d(Tx_n,By_n)},$$

$$d(Tx_n,Sy_n), d(Tx_n,Ax_n), d(By_n,Sy_n)$$

$$d(Tx_n,By_n), d(Sy_n,Ax_n), \frac{d(By_n,Sy_n)\,d(Ax_n,Sy_n)}{d(Tx_n,By_n)}\}$$

Taking limit as n→∞ in (11) we get

$$\lim_{n\to\infty} \int_0^{d(Ax_n,By_n)} \phi(t)\,dt < \lim_{n\to\infty} \int_0^{M(x_n,y_n)} \phi(t)\,dt, \quad (12)$$

Since,

$$\lim_{n\to\infty} d(Tx_n,Sy_n) = \lim_{n\to\infty} d(Tx_n,Ax_n) = \lim_{n\to\infty} d(Sy_n,Ax_n) = 0$$

$$\lim_{n\to\infty} d(Ax_n,By_n) = \lim_{n\to\infty} d(Ax,By_n) = \lim_{n\to\infty} d(By_n,Sy_n)$$

Hence we have

$$\lim_{n\to\infty} \int_0^{d(Ax,By_n)} \phi(t)\,dt \le 0$$

which is a contradiction.

Therefore,

$$\lim_{n\to\infty} d(Ax,By_n) = 0 \Rightarrow \lim_{n\to\infty} By_n = Ax.$$

Now we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = Ax$$

Assume $A(X) \subseteq S(X)$, then there exits $v \in X$ such that $Ax=Sv$.

We claim that Bv=Sv.

Now from relation (6)

$$\int_0^{d(Ax_n,Bv)} \phi(t)\,dt < \int_0^{M(x_n,v)} \phi(t)\,dt \quad (13)$$

$$M(x_n,v) = Min\{\frac{d(Tx_n,Sv)\,d(Tx_n,Ax_n)}{d(Tx_n,Bv)}, d(Tx_n,Sv), d(Tx_n,Ax_n), d(Bv,Sv),$$

where,

$$d(Tx_n,Bv), d(Sv,Ax_n), \frac{d(Bv,Sv)\,d(Ax_n,Sv)}{d(Tx_n,Bv)}\}$$

Since,

$$\lim_{n\to\infty} d(Tx_n,Bv) = d(Ax,Bv) = d(Sv,Bv)$$

$$\lim_{n\to\infty} d(Tx_n,Sv) = \lim_{n\to\infty} d(Tx_n,Ax_n) = \lim_{n\to\infty} d(Sv,Ax_n) = 0$$

So, taking the limit as n→∞ in (13), we conclude that

$$\int_0^{d(Sv,Bv)} \phi(t)\,dt < 0 \quad (14)$$

which is a contradiction.

Hence $d(Sv,Bv)=0 \Rightarrow Sv=Bv$.
It proves that v is the coincidence point of of the maps B and S .

Therefore,

$Sv=Bv=Ax=w$ (Say)
Since the pair (B,S) is weakly compatible, so
$BSv=SBv \Rightarrow Bw=Sw$

Since $B(X) \subseteq T(X)$ there exists a point $u \in X$ such that $Bv=Tu$. We show that $Tu=Au=w$

From relation (6),

$$\int_0^{d(Au,Bv)} \phi(t)\,dt < \int_0^{M(u,v)} \phi(t)\,dt,$$

where,

$$M(u,v) = Min\{\frac{d(Tu,Sv)\,d(Tu,Au)}{d(Tu,Bv)}, d(Tu,Sv), d(Tu,Au), d(Bv,Sv),$$

$$d(Tu,Bv), d(Sv,Au), \frac{d(Bv,Sv)\,d(Au,Sv)}{d(Tu,Bv)}\}$$

$=Min\{d(Bv,Au),d(Bv,Bv),d(Bv,Au),d(Bv,Bv), d(Bv,Bv),d(Bv,Au),d(Au,Bv)\}$
$=Min\{d(Bv,Bv),d(Bv,Au)\}$
$=d(Bv,Au)$

Since $d(Bv,Bv) \leq 2d(Bv,Au)$

Hence,

$$\therefore \int_0^{d(Au,Bv)} \phi(t)\,dt < \int_0^{M(u,v)} \phi(t)\,dt = \int_0^{d(Au,Bv)} \phi(t)\,dt$$

which is a contradiction. Hence $d(Au,Bv) = 0 \Rightarrow Au=Bv$.

$\therefore\ Au=Bv=Tu=w$

This proves that u is the coincidence point of the maps A and T.

Since the pair (A,T) is weakly compatible so,

$ATu=TAu \Rightarrow Aw=Tw$

We show that $Aw=w$.

From relation *(6)*

$$\int_0^{d(Aw,w)} \phi(t)\,dt = \int_0^{d(Aw,Bv)} \phi(t)\,dt < \int_0^{M(w,v)} \phi(t)\,dt,$$

where,

$$M(w,v) = Min\{\frac{d(Tw,Sv)\,d(Tw,Aw)}{d(Tw,Bv)}, d(Tw,Sv), d(Tw,Aw), d(Bv,Sv),$$

$$d(Tw,Bv), d(Sv,Aw), \frac{d(Bv,Sv)\,d(Aw,Sv)}{d(Tw,Bv)}\}$$

$$= Min\{\frac{d(Aw,w)\,d(Aw,Aw)}{d(Aw,w)}, d(Aw,w), d(Aw,Aw), d(w,w),$$

$$d(Aw,w), d(w,Aw), \frac{d(w,w)\,d(Aw,w)}{d(Aw,w)}\}$$

$=Min\{d(Aw,w),d(Aw,Aw),d(w,w)\}$
$=d(Aw,w)$

$$\therefore \int_0^{d(Aw,w)} \phi(t)\,dt = \int_0^{d(Aw,Bv)} \phi(t)\,dt < \int_0^{M(w,v)} \phi(t)\,dt = \int_0^{d(Aw,w)} \phi(t)\,dt$$

which is a contradiction.

Hence $d(Aw,w)= 0 \Rightarrow Aw=w$. Similarly we obtain $Bw=w$.

$\therefore\ Aw = Bw = Sw = Tw = w$. It establishes that w is the common fixed point of four mappings A,B,S and T.

Uniqueness

let $z(\neq w)$ be another common fixed point of the mappings A,B,S *and* T, then by the relation (6)

$$\int_0^{d(w,z)} \phi(t)\,dt = \int_0^{d(Aw,Bz)} \phi(t)\,dt < \int_0^{M(w,z)} \phi(t)\,dt \quad (15)$$

where,

$$M(w,z) = Min\{\frac{d(Tw,Sz)\,d(Tw,Aw)}{d(Tw,Bz)}, d(Tw,Sz), d(Tw,Aw), d(Bz,Sz)$$

$$d(Tw,Bz), d(Sz,Aw), \frac{d(Bz,Sz)\,d(Aw,Sz)}{d(Tw,Bz)}\}$$

$$= Min\{\frac{d(w,z)\,d(w,w)}{d(w,z)}, d(w,z), d(w,w), d(z,z),$$

$$d(w,z), d(z,w), \frac{d(z,z)\,d(w,z)}{d(w,z)}\}$$

$=Min\{d(w,z),d(w,w),d(z,z)\}$
$=d(w,z)$

$$\therefore \int_0^{d(w,z)} \phi(t)\,dt = \int_0^{d(Aw,Bz)} \phi(t)\,dt < \int_0^{M(w\,z)} \phi(t)\,dt = \int_0^{d(w,z)} \phi(t)\,dt$$

which is a contradiction.

Hence, $d\ (w,z) = 0 \Rightarrow w = z$. This establishes the uniqueness of the common fixed point.

We can obtain the following corollaries with the help of above theorem.

Corollary 1 *Let (X,d) be a dislocated metric space. Let A,B,S:X→X satisfying the following conditions A(X)* and *B(X) ⊆ S(X)*

$$\int_0^{d(Ax,By)} \phi(t)\,dt < \int_0^{M(x,y)} \phi(t)\,dt)$$

where,

$$M(x,y) = Min\{\frac{d(Sx,Sy)\,d(Sx,Ax)}{d(Sx,By)}, d(Sx,Sy), d(Sx,Ax), d(By,Sy)$$

$$d(Sx,By), d(Sy,Ax), \frac{d(By,Sy)\,d(Ax,Sy)}{d(Sx,By)}\}$$

$$\phi : \mathbb{R}^+ \to \mathbb{R}^+$$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad for\ each\ \epsilon > 0$$

1. The pairs (*A,S*) or (*B,S*) satisfy (CLR)-property
2. The pairs (*A,S*) and (*B,S*) are weakly compatible

then the pairs (A, S) and (B, S) have coincidence point and the three maps A,B and S have an unique common fixed point.

Corollary 2 *Let (X,d) be a dislocated metric space. Let A,S,T:X→X satisfying the following conditions A(X)⊆ S(X) and A(X) ⊆ T(X)*

$$\int_0^{d(Ax,Ay)} \phi(t)\,dt < \int_0^{M(x,y)} \phi(t)\,dt)$$

where,

$$M(x,y) = Min\{\frac{d(Sx,Sy)\,d(Sx,Ax)}{d(Sx,Ay)}, d(Sx,Sy), d(Sx,Ax), d(Ay,Sy)$$

$$d(Sx,Ay), d(Sy,Ax), \frac{d(Ay,Sy)\,d(Ax,Sy)}{d(Sx,Ay)}\}$$

$$\phi : \mathbb{R}^+ \to \mathbb{R}^+$$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad for\ each\ \epsilon > 0$$

1. The pairs *(A,T)* or *(A,S)* satisfy (CLR)-property
2. The pairs *(A,T)* and *(A,S)* are weakly compatible

then the pairs *(A, T)* and *(A, S)* have coincidence point and the three maps A,S and T have an unique common fixed point.

Corollary 3 *Let (X,d) be a dislocated metric space. Let A,S:X→X satisfying the following conditions A(X) ⊆ S(X)*

$$\int_0^{d(Ax,Ay)} \phi(t)\,dt < \int_0^{M(x,y)} \phi(t)\,dt)$$

where,

$$M(x,y) = Min\{\frac{d(Sx,Sy)\,d(Sx,Ax)}{d(Sx,Ay)}, d(Sx,Sy), d(Sx,Ax), d(Ay,Sy)$$

$$d(Sx,Ay), d(Sy,Ax), \frac{d(Ay,Sy)\,d(Ax,Sy)}{d(Sx,Ay)}\}$$

$$\phi : \mathbb{R}^+ \to \mathbb{R}^+$$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad for\ each\ \epsilon > 0$$

1. The pairs (A,S) satisfy (CLR)-property
2. The pairs (A,S) is weakly compatible

then the pair (A, S) have a coincidence point and an unique common fixed point.

Corollary 4 *Let (X,d) be a dislocated metric space. Let A,B, I : X → X satisfying the following conditions A(X),B(X) ⊆ I(X)*

$$\int_0^{d(Ax,By)} \phi(t)\,dt < \int_0^{M(x,y)} \phi(t)\,dt)$$

where,

$$M(x,y) = Min\{\frac{d(x,y)d(x,Ax)}{d(x,By)}, d(x,y), d(x,Ax), d(By,Sy)$$

$$d(x,By), d(y,Ax), \frac{d(By,y)d(Ax,y)}{d(x,By)}\}$$

$$\phi : \mathbb{R}^+ \to \mathbb{R}^+$$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad for\ each\ \epsilon > 0$$

1. The pairs (*A, I*) or (*B, I*) satisfy (CLR)-property
2. The pairs (*A, I*) and (*B, I*) are weakly compatible then the pairs (*A, I*) and (*B, I*) have coincidence point and the three maps A,B and I have an unique common fixed point.

We establish a common fixed point theorem satisfying integral and rational type contractive condition for two pairs of compatible maps with reciprocally continuous property in metric space.

Theorem 5 *Let (X,d) be a complete metric space. Let A, B, S, T :X → X such that the pairs (A, S) and (B, T) be compatible mappings which satisfy the following conditions*

$$A(X) \subset T(X), B(X) \subset S(X) \qquad (16)$$

Given, $\varepsilon > 0, \exists \delta > 0,$ such that

$$\varepsilon < M(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) \le \varepsilon$$

where,

$$M(x,y) = Max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}\left[d(Ax,Ty) + d(By,Sx)\right]\right\}$$

$$d(Ax,By) < \phi(k\{\frac{d(Sx,Ty)d(Sx,Ax)}{d(Sx,By)} + d(Sx,Ty) + d(Ax,Sx) \quad (17)$$

$$+d(By,Ty) + d(Ax,Ty) + d(By,Sx) + \frac{d(By,Ty)d(Ax,Ty)}{d(Sx,By)}\})$$

$k \in \left(0, \frac{1}{3}\right]$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\phi(t) < t$ for some $t > 0$.

Suppose that the mappings in one of the pairs (*A, S*) or (*B, T*) are reciprocally continuous, then *A, B, S* and *T* have an unique common fixed point.

Proof:

let $x_0 \in X$ be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \qquad (18)$$

$$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}\ for\ n \in \mathbb{N} \cup \{0\} \qquad (19)$$

then by Jachymski's lemma 1, $\{y_n\}$ is a Cauchy sequence. Sine X is complete, so there exists a point $z \in X$ such that $y_n \to z$. Also the sequences

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z\ and\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z \qquad (20)$$

Suppose that the pair *(A, S)* is reciprocally continuous, then $ASx_n \to Az$ and $SAx_n \to Sz$. Since the pair *(A, S)* is compatible so

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n} = z\ and\ \lim_{n\to\infty}\left(ASx_{2n}, SAx_{2n}\right) = 0 \qquad (21)$$

This implies that $d(Az, Sz) = 0$. Hence $Az = Sz$.
Since, $A(X) \subset T(X)$ there exists a point $w \in X$ such that $Az = Tw$.
We claim that $Bw = Tw$.
If $Bw \neq Tw$, then from relation (17) we get,

$$d(Az,Bw) < \phi(k\{\frac{d(Sz,Tw)d(Sz,Az)}{d(Sz,Bw)} + d(Sz,Tw) + d(Az,Sz)$$

$$+d(Bw,Tw) + d(Az,Tw) + d(Bw,Sz) + \frac{d(Bw,Tw)d(Az,Tw)}{d(Sz,Bw)}\})$$

$= \phi(k\{d(Bw, Az) + d(Bw, Az)\})$
$=\phi(2kd(Az, Bw))$
$=2k\phi(d(Az, Bw))$
$<2kd(Az, Bw)$

which is a contradiction. So $d(Az, Bw) = 0 \Rightarrow Az = Bw$. Hence $Az = Sz = Tw = Bw$.

Since the compatible maps commute at their coincidence point, we get $ASz = SAz$. This further implies that $AAz = ASz = SAz = SSz$.

If $Az \neq AAz$, now by relation (17) we get

$d(Az,AAz)=d(Bw,AAz)=d(AAz,Bw)$

$$< \phi(k\{\frac{d(SAz,Tw)d(SAz,AAz)}{d(SAz,Bw)} + d(SAz,Tw) + d(AAz,SAz)$$

$$d\left(Bw,Tw\right)+d\left(AAz,Tw\right)+d\left(Bw,SAz\right)+\frac{d\left(Bw,Tw\right)d\left(AAz,Tw\right)}{d\left(SAz,Bw\right)}\})$$

$= \phi(k\{d(AAz, Bw) + d(AAz, Bw)+d(Bw, AAz)\})$

$=\phi(3kd(AAz, Bw))$

$<3kd(AAz, Az)$

which is a contradictions, so $d(Az, AAz) = 0 \Rightarrow Az = AAz$. Hence, $Az = AAz = SAz$. Thus Az is the common fixed point of the mappings A and S. Similarly we obtain $Bw(=Az)$ is the common fixed point of the mappings B and T.

Uniqueness

If possible, let u and $v(u \neq v)$ are two common fixed points of the maps A, B, S and T. Now by virtue of relation (17)

$$d\left(u,v\right)=d\left(Au,Bv\right)<\phi(k\{\frac{d\left(Su,Tv\right)d\left(Su,Au\right)}{d\left(Su,Bv\right)}+d\left(Su,Tv\right)+d\left(Au,Su\right)$$

$$+d\left(Bv,Tv\right)+d\left(Au,Tv\right)+d\left(Bv,Su\right)+\frac{d\left(Bv,Tv\right)d\left(Au,Tv\right)}{d\left(Su,Bv\right)}\})$$

$$=\phi(k\{\frac{d\left(u,v\right)d\left(u,u\right)}{d\left(u,v\right)}+d\left(u,v\right)+d\left(u,u\right)$$

$$+d\left(v,v\right)+d\left(u,v\right)+d\left(v,u\right)+\frac{d\left(v,v\right)d\left(u,v\right)}{d\left(u,v\right)}\})$$

$= \phi(3kd(u, v))$

$< 3kd(u, v)$

which is a contradiction. This shows that $d(u,v)=0 \Rightarrow u=v$.

The proof is similar when the mappings B and T are assumed compatible and reciprocally continuous. It completes the proof of the theorem.

We can establish the following corollaries with the help of the above theorem.

Corollary 5 *Let (X,d) be a complete metric space. Let A, B, S: X → X such that the pairs (A, S) and (B, S) be compatible mappings which satisfy the following conditions*

$A(X)$ and $B(X) \subset S(X)$

Given, $\varepsilon > 0, \exists\delta > 0$, such that

$$\varepsilon < M\left(x,y\right) < \epsilon + \delta \Rightarrow d\left(Ax,By\right) \le \varepsilon$$

$$M\left(x,y\right)=Max\left\{d\left(Sx,Sy\right),d\left(Ax,Sx\right),d\left(By,Sy\right),\frac{1}{2}\left[d\left(Ax,Sy\right)+d\left(By,Sx\right)\right]\right\}$$

$$d\left(Ax,By\right)<\phi(k\{\frac{d\left(Sx,Sy\right)d\left(Sx,Ax\right)}{d\left(Sx,By\right)}+d\left(Sx,Sy\right)+d\left(Ax,Sx\right)$$

$$+d\left(By,Sy\right)+d\left(Ax,Sy\right)+d\left(By,Sx\right)+\frac{d\left(By,Sy\right)d\left(Ax,Sy\right)}{d\left(Sx,By\right)}\})$$

$k \in \left(0,\frac{1}{3}\right]$ $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\phi(t) < t$ for some $t > 0$.

Suppose that the mappings in one of the pairs *(A, S)* or *(B, S)* are reciprocally continuous, then the three maps A, B and S have a unique common fixed point.

Corollary 6 *Let (X,d) be a complete metric space. Let A, S, T: X → X such that the pairs (A, S) and (B, T) be compatible mappings which satisfy the following conditions*

$A(X) \subset T(X), A(X) \subset S(X)$

Given, $\varepsilon > 0, \exists\delta > 0$, *such that*

$$\varepsilon < M\left(x,y\right) < \epsilon + \delta \Rightarrow d\left(Ax,Ay\right) \le \varepsilon$$

where,

$$M\left(x,y\right)=Max\left\{d\left(Sx,Ty\right),d\left(Ax,Sx\right),d\left(Ay,Ty\right),\frac{1}{2}\left[d\left(Ax,Ty\right)+d\left(Ay,Sx\right)\right]\right\}$$

$$d\left(Ax,Ay\right)<\phi(k\{\frac{d\left(Sx,Ty\right)d\left(Sx,Ax\right)}{d\left(Sx,Ay\right)}+d\left(Sx,Ty\right)+d\left(Ax,Sx\right)$$

$k \in \left(0,\frac{1}{3}\right]$ $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\phi(t) < t$ for some $t > 0$.

Suppose that the mappings in one of the pairs *(A, S)* or *(A, T)* are reciprocally continuous, then the three maps A, S and T have an unique common fixed point.

Corollary 7 *Let (X, d) be a complete metric space. Let A, S : X → X such that the pairs (A, S) be compatible mappings which satisfy the following conditions*

A(X) ⊂ *S(X)*

Given, $\varepsilon > 0, \exists\delta > 0$, such that

$$\varepsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, Ay) \leq \varepsilon$$

where,

$$M(x,y) = Max\left\{d(Sx,Ty), d(Ax,Sx), d(Ay,Ty), \frac{1}{2}\left[d(Ax,Ty) + d(Ay,Sx)\right]\right\}$$

$$d(Ax, Ay) < \phi(k\{\frac{d(Sx,Ty)\,d(Sx,Ax)}{d(Sx,Ay)} + d(Sx,Ty) + d(Ax,Sx)$$

$$+d(Ay,Ty) + d(Ax,Ty) + d(Ay,Sx) + \frac{d(Ay,Ty)\,d(Ax,Ty)}{d(Sx,Ay)}\})$$

$\kappa \in \left(0, \frac{1}{3}\right]$ $\phi : \mathbb{R}^{+} \to \mathbb{R}^{+}$is such that $\phi(t) < t$ for some $t > 0$.

Suppose that the pair *(A, S)* is reciprocally continuous, then the pair have a unique common fixed point.

Corollary 8 *Let (X, d) be a complete metric space. Let A, B, I : X → X (I is the identity mapping) such that the pairs (A, I) and (B, I) be compatible mappings which satisfy the following conditions*

A(X) and B(X) ⊂ *I(X)*

Given, $\varepsilon > 0, \exists\delta > 0$, such that

$$\varepsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, Ay) \leq \varepsilon$$

$$M(x,y) = max\left\{d(x,y), d(Ax,x), d(By,y), \frac{1}{2}\left[d(Ax,y) + d(By,x)\right]\right\}$$

$$d(Ax, By) < \phi(k\{\frac{d(x,y)\,d(x,Ax)}{d(x,By)} + d(x,y) + d(Ax,x)$$

$$+d(By,y) + d(Ax,y) + d(By,x) + \frac{d(By,y)\,d(Ax,y)}{d(x,By)}\})$$

$\kappa \in \left(0, \frac{1}{3}\right]$ $\phi : \mathbb{R}^{+} \to \mathbb{R}^{+}$is such that $\phi(t) < t$ for some $t > 0$.

Suppose that the mappings in one of the pairs *(A, I)* or *(B, I)* are reciprocally continuous, then *A, B* and *I* have an unique common fixed point.

4. CONCLUSION

In recent works, use of Common limit Range (CLR) property for rational and integral type contractive condition are established on complexvalued metric space (Chandok & Kumar 2013), metric space (Nashine 2015) partially ordered metric space (Rao & Vijaya Sri 2019), b-metric spaces (Seddik & Taieb 2021) and more. We have established some new common fixed point results in dislocated metric space. (CLR) property does not require conditions like continuity of one or more maps, closedness of subspaces. So this property is supposed to be more interesting for the establishment of common fixed point results in different types of generalized metric spaces in the literature. In the second theorem (Theorem 5), we have established a common fixed point result for rational and integral type contractive conditions in dislocated metric space with compatible and reciprocal continuity of mappings.

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