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On Certain Topological Properties of Normed Space Valued Null Function Space $c_0(S, (E, \|\cdot\|), \xi, u)$ and Its Paranormed Structure

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Abstract
The aim of this paper is to introduce and study a new class $c_0(S, (E, \|\cdot\|), \xi, u)$ of normed space E valued functions which will generalize some of the well known basic sequence spaces and function spaces studied in Functional Analysis. Beside the investigation pertaining to the linear paranormed structure of the class $c_0(S, (E, \|\cdot\|), \xi, u)$ when topologized it with suitable natural paranorm , our primarily interest is to explore the conditions pertaining the containment relation of the class $c_0(S, (E, \|\cdot\|), \xi, u)$ in terms of different ξ and u so that such a class of functions is contained in or equal to another class of similar nature.

Key words: paranormed space, normed space, summable family

Introduction
We begin with recalling some notations and basic definitions that are used in this paper.

The concept of paranormed space is closely related to linear metric space, see Wilansky (1978) and its studies on sequence spaces were initiated by Maddox (1969) and many others.

Definition 1. Let S be the linear space with zero element θ over $\mathbb{C}$. A paranormed space (S, G) is a linear space S together with a function $G : S \to \boldsymbol{R}^+$ (called a paranorm on S) which satisfies the following axioms:

PN_1: $G(\theta) = 0$;

PN_2: $G(x) = G(-x)$ for all $x \in S$;

PN_3: $G(x_1 + x_2) \le G(x_1) + G(x_2)$ for all $x_1, x_2 \in S$, and

PN_4: if $< \alpha_n >$ be a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and $< x_n >$ be a sequence in S with $G(x_n - x) \to 0$ as $n \to \infty$, then $G(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$ (continuity of scalar multiplication).

Note that the continuity of scalar multiplication is equivalent to

(i) if $G(x_n) \to 0$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $G(\alpha_n x_n) \to 0$ as $n \to \infty$; and

(ii) if $\alpha_n \to 0$ as $n \to \infty$ and x be any element in S, then $G(\alpha_n x) \to 0$, see Wilansky (1978).

A paranorm is called total if in addition we have

NP_5: $G(x) = 0$ implies $x = \theta$.

Nanda *et al* (1983), Parasar and Choudhary (1994), Bektas and Altin (2003) , Bhardwaj and Bala (2007), Khan (2008), Basariv and Altundag (2009) and many others further studied various types of paranormed sequence spaces .

Definition 2. A normed space $(S, \|.\|)$ is a linear space S with zero element θ together with the mapping $\|.\| : S \to \boldsymbol{R}^+$ (called *norm* on S) such that for all $x, x_1, x_2 \in S$ and $\alpha \in \mathbb{C}$, we have

N_1: $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = \theta$;

N_2: $\|\alpha x\| = |\alpha| \|x\|$; and

N_3: $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$.

Clearly by N_1 and N_2, the algebraic operations of addition and scalar multiplication in the normed space S are continuous i.e., if $< x_n >$ and $< x'_n >$ are sequences in the normed space S with $x, x' \in S$ such that $x_n \to x$, $x'_n \to x'$ in S, and $< \alpha_n >$ a sequence of scalars such that $\alpha_n \to \alpha$ in C, then $x_n + x'_n \to x + x'$ and $\alpha_n x_n \to \alpha x$ in S

In fact, Maddox (1969), Kamthan and Gupta (1981), Srivastava (1996), Tiwari and Srivastava (2008, 2010),Srivastava and Pahari (2011, 2011, 2013) and Pahari (2011,2013,2014) and many others have been introduced and studied the algebraic and topological properties of various sequence and function spaces in normed spaces. All these sequence and function spaces generalize and unify various existing basic sequence spaces studied in Functional Analysis.

Definition 3. For the set of indices S, $\mathcal{F}(S)$ denotes the collection of all finite subsets J of S. The set $\mathcal{F}(S)$ is an ordered set with respect to set theoretic inclusion relation $\subset$ used as $\leq$.

If $\alpha: S \to C$, then the family $< \alpha(x) >_{x \in S}$ is said to converge to $\ell \in C$ if for each positive number ε there exists a set $J \in \mathcal{F}(S)$ such that $|\alpha(x) - \ell| < \varepsilon$, for all $x \in S - J$. Definition 4. Let $\alpha: S \to C$ and $< \alpha(x) >_{x \in S}$, be the family of numbers. Then the finite partial sums

$$P_J = \sum_{x \in J} \alpha(x),\ J \in \mathcal{F}(S),$$

form a directed system with respect to set theoretic inclusion relation . If this directed system $< P_J >_{J \in \mathcal{F}(S)}$ converges then we say that the family is summable. The uniquely determined limit ℓ is then called the sum of the family and we write $\ell = \sum_{x \in J} \alpha(x)$. If the index set is finite then ℓ obviously coincides with the ordinary sum.

The class $c_0(S, (E, \|.\|), \xi, u)$ of normed space valued functions

Let S be an arbitrary non empty set (not necessarily countable) and $\mathcal{F}(S)$ be the collection of all finite subsets of S directed by inclusion relation. Let $(E, \|.\|)$ be a normed space over the field of complex numbers C. We shall write u, v for functions on $S \to R^+$, the set of all positive real numbers, and

$\ell_\infty(S, R^+) = \{ u : S \to R^+$ such that $\sup_x u(x) < \infty \}$.

Further we write ξ, μ for functions on $S \to C - \{0\}$ and collection of all such functions will be denoted by $s(S, C - \{0\})$.We shall also frequently use the notations

$0 < \sup_{x \in S} u(x) = L < \infty$ and for scalar α,

$A[\alpha] = \max(1, |\alpha|)$.

But when the functions $u(x)$ and $v(x)$ occur, then to distinguish L we use the notations $L(u)$ and $L(v)$ respectively.

We now introduce the following new class of normed space E- valued functions:

$c_0(S, (E, \|.\|), \xi, u) = \{ \phi : S \to E$: for every $\varepsilon > 0$, there exists a finite subset $J \in \mathcal{F}(S)$ such that $\| \xi(x)\ \phi(x) \|^{u(x)} < \varepsilon$, for each $x \in S - J \}$.

Further when $\xi : S \to C - \{0\}$ is a function such that $\xi(x) = 1$ for all x, then $c_0(S, (E, \|.\|), \xi, u)$ will be denoted by $c_0(S, (E, \|.\|), u)$ and when $u : S \to R^+$ is a function such that $u(x) = 1$ for all x, then $c_0(S, (E, \|.\|), \xi, u)$ will be denoted by $c_0(S, (E, \|.\|), \xi)$.

Actually, this class is the generalization of the familiar sequence and function spaces, studied in Srivastava *et al.* (1996), Srivastava (1996), Tiwari *et al.* (2008, 2010), Srivastava and Pahari (2011, 2011, 2013), Pahari (2011, 2013,2014).

As far as the linear space structure of this class over the field C of complex numbers is concerned, we throughout take pointwise operations i.e.,for functions ϕ, ψ and scalar α,

$(\phi + \psi)(x) = \phi(x) + \psi(x)$ and $(\alpha\,\phi)(x) = \alpha\,\phi(x), x \in S$

and we see below that if $u \in \ell_\infty(S, R^+)$, then the class $c_0(S, (E, \|.\|), \xi, u)$ forms a linear space over C. Moreover, we shall denote the zero element of this space by θ by which we shall mean the function $\theta : S \to E$ such that $\theta(x) = \theta$, for all $x \in S$.

Main Results on $c_0(S, (E, \|.\|), \xi, u)$

In this section, we first study the linear paranorme of the class $c_0(S, (E, \|.\|), \xi, u)$ by endow: paranorm arising in a natural way and then study the $(E, \|.\|), \xi, u)$ of normed space E- valued functioi of different u and ξ so that such a class is contained to another class of similar nature.Throughout the woi denote

$z(x) = \frac{v(x)}{u(x)}$, $w(x) = \left|\frac{\xi(x)}{\mu(x)}\right|^{u(x)}$.

Moreover, we use

$|a + b|^{u(x)} \le T[|a|^{u(x)} + |b|^{u(x)}]$, where $a, b \in \mathbf{C}$ and $T = \max(1, 2^{L-1}) = A[2^{L-1}]$.

Theorem 1. The set $c_0(S, (E, \|\cdot\|), \xi, u)$ forms a linear space over the field $\boldsymbol{C}$ of complex numbers with respect to the pointwise vector operations if $< z(x) >_{x \in S}$ is bounded above.

Proof:

Suppose $\sup_{x \in S} u(x) < \infty$ and $\phi, \psi \in c_0(S, (E, \|\cdot\|), \xi, u)$. Let $\varepsilon > 0$ be given. Then there exist finite subsets $J_1, J_2 \in \mathcal{F}(S)$ such that

$$\|\xi(x)\,\phi(x)\|^{u(x)} < \frac{\varepsilon}{2}, \quad \text{for every } x \in S - J_1 \qquad (1)$$

and

$$\|\xi(x)\,\psi(x)\|^{u(x)} < \frac{\varepsilon}{2}, \quad \text{for every } x \in S - J_2. \qquad (2)$$

Let $J = J_1 \cup J_2$ and for each $x \in S - J$ and in view of (1) and (2), we have

$$\begin{aligned}
&\|\xi(x)[\phi(x) + \psi(x)]\|^{u(x)} \\
&\le \|\xi(x)\,\phi(x) + \xi(x)\,\psi(x)\|^{u(x)} \\
&\le T[\|\xi(x)\,\phi(x)\|^{u(x)} + \|\xi(x)\,\psi(x)\|^{u(x)}] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which clearly shows that $\phi+\psi \in c_0(S, (E, \|.\|), \xi, u)$.

Let $\alpha \in \boldsymbol{C}$, then we have

$$\begin{aligned}
\|\alpha\,\xi(x)\,\phi(x)\|^{u(x)} &\le |\alpha|^{u(x)}\,\|\xi(x)\,\phi(x)\|^{u(x)} \\
&\le A[\alpha]\,\|\xi(x)\,\phi(x)\|^{u(x)}, \; x \in S - J.
\end{aligned}$$

This clearly gives us that $\alpha\phi \in c_0(S, (E, \|.\|), \xi, u)$. Thus if $\sup_{x \in S} u(x) < \infty$ then $c_0(S, (E, \|\cdot\|), \xi, u)$ forms a linear space over the field C. This completes the proof.

Now for $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$, define a real valued function G by

$$G(\phi) = \sup_x \|\xi(x)\,\phi(x)\|^{u(x)/L} \qquad (3)$$

Theorem 2. The space $c_0(S, (E, \|\cdot\|), \xi, u)$ forms a paranormed space with respect to G defined by (3).

Proof:

For $\phi, \psi \in c_0(S, (E, \|\cdot\|), \xi, u)$, $G(\phi) \ge 0$, $G(\theta) = 0$, $G(-\phi) = G(\phi)$ and $G(\phi + \psi) \le G(\phi) + G(\psi)$ can easily be proved.

So PN_1, PN_2 and PN_3 are obvious.

We only prove below PN_4 i.e., the continuity of scalar multiplication.

(i) Let $< \phi_k >$ be a sequence in $c_0(S, (E, \|\cdot\|), \xi, u)$ and $< \alpha_k >$ a sequence of scalars such that $G(\phi_k) \to 0$ and $\alpha_k \to \alpha$ as $k \to \infty$.

Suppose that $|\alpha_k| \le 1$ for $k \ge 1$. Then we have

$$\begin{aligned}
G(\alpha_k\,\phi_k) &= \sup_x \|\alpha_k\,\xi(x)\,\phi_k(x)\|^{u(x)/L} \\
&= \sup_x |\alpha_k|^{u(x)/L}\,\sup_x \|\xi(x)\,\phi_k(x)\|^{u(x)/L} \\
&\le \sup_x |M|^{u(x)/L}\,\sup_x \|\xi(x)\,\phi_k(x)\|^{u(x)/L} \\
&\le A[M]\,G(\phi_k)
\end{aligned}$$

implies that $G(\alpha_k\,\phi_k) \to 0$ as $k \to \infty$.

(ii) Let $\alpha_k \to 0$ as $k \to \infty$. Without loss of generality we can assume that $|\alpha_k| \le 1$, $k \ge 1$. Let $\varepsilon > 0$ be given. Then for $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$ there exists a finite subset J of S such that

$\|\xi(x)\,\phi(x)\|^{u(x)} < \varepsilon$, for all $x \in S - J$.

In particular, $\|\xi(x)\,\phi(x)\|^{u(x)/L} < \varepsilon$, for all $x \in S - J$.

Thus,

$$\begin{aligned}
\|\alpha_k\,\xi(x)\,\phi(x)\|^{u(x)/L} &= |\alpha_k|^{u(x)/L}\,\|\xi(x)\,\phi(x)\|^{u(x)/L} \\
&\le \|\xi(x)\,\phi(x)\|^{u(x)/L} \\
&< \varepsilon, \text{ for all } x \in S - J.
\end{aligned}$$

Now choose a positive integer k_0 such that for any $x \in J$

$$\begin{aligned}
G(\alpha_k\,\phi) &= \|\alpha_k\,\xi(x)\,\phi(x)\|^{u(x)/L} \\
&= |\alpha_k|^{u(x)/L}\,\|\xi(x)\,\phi(x)\|^{u(x)/L} \\
&< \varepsilon, \text{ for all } k \ge k_0.
\end{aligned}$$

Hence, $G(\alpha_k\,\phi) < \varepsilon$ for all $k \ge k_0$. This means that $G(\alpha_k\,\phi) \to 0$ as $k \to \infty$.

Thus (i) and (ii) prove the continuity of scalar multiplication. Hence G forms a paranorm on $c_0(S, (E, \|\cdot\|), \xi, u)$ and it completes the proof.

Theorem 3. If $\sup_{x \in S} u(x) < \infty$, $v : S \to \boldsymbol{R}^+$ and $\xi \in s(S, \boldsymbol{C} - \{0\})$, then

$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \xi, v)$

if and only if $< z(x) >_{x \in S}$ has positive limit inferior.

Proof:

For the sufficiency of the condition, assume that $\lim\inf_x z(x) > 0$. Then there exists $m > 0$ such that $v(x) > m\,u(x)$, for all but finitely many $x \in S$.

Let $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$ and $\varepsilon > 0$ be given. Then for $0 < \eta < 1$ with $\eta^m < \varepsilon$, there exists $J \in \mathcal{F}(S)$ satisfying

$\|\xi(x)\,\phi(x)\|^{u(x)} < \eta$ and so

$$\begin{aligned}
\|\xi(x)\,\phi(x)\|^{v(x)} &\le [\|\xi(x)\,\phi(x)\|^{u(x)}]^m \\
&< \eta^m < \varepsilon, \text{ for each } x \in S - J.
\end{aligned}$$

Thus we easily get $\phi \in c_0(S, (E, \|\cdot\|), \xi, v)$ and hence

$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \xi, v)$.

For the necessity of the condition, assume that $c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \xi, v)$ but $\liminf_x z(x) = 0$. Then there exists a sequence $< x_k >$ of distinct points in S such that for each

$$k \geq 1, \ k\, v(x_k) < u(x_k) \qquad (4)$$

Now, taking $e \in E$ with $\| e \| = 1$, define $\phi : S \to E$ by the function

$$\phi(x) = \begin{cases} (\xi(x))^{-1} k^{-1/u(x)} e, \text{ for } x = x_k, k = 1, 2, 3, \ldots, \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \qquad (5)$$

Then for each $k \geq 1$, we have

$$\| \xi(x_k)\ \phi(x_k) \|^{u(x_k)} = \| k^{-1/u(x_k)} e \|^{u(x_k)}$$
$$= \frac{1}{k} \| e \|^{u(x_k)}$$
$$\leq \frac{1}{k} A [\| e \|^{I(u)}]$$

and for $x \neq x_k$, $k \geq 1$, $\| \xi(x)\ \phi(x) \|^{u(x)} = 0$.

This shows that $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$. But on the other hand for each $k \geq 1$, in view of (4) and (5) we have

$$\| \xi(x_k)\ \phi(x_k) \|^{v(x_k)} = \| k^{-1/u(x_k)} e \|^{v(x_k)}$$
$$= \frac{1}{k^{v(x_k)/u(x_k)}} \| e \|^{v(x_k)}$$
$$> \frac{1}{k^{1/k}}$$
$$\geq \frac{1}{e^{1/2}} .$$

This shows that $\phi \notin c_0(S, (E, \|\cdot\|), \xi, v)$, a contradiction. This completes the proof.

Theorem 4. Let $u : S \to \mathbf{R}^+$, $v \in \ell_\infty(S, \mathbf{R}^+)$ and $\xi \in s(S, \mathbf{C} - \{0\})$, then

$c_0(S, (E, \|\cdot\|), \xi, v) \subset c_0(S, (E, \|\cdot\|), \xi, u)$

if and only if $< z(x) >_{x \in S}$ has finite limit superior.

Proof:

For the necessity of the condition, suppose that $c_0(S, (E, \|\cdot\|), \xi, v) \subset c_0(S, (E, \|\cdot\|), \xi, u)$ but $\limsup_x z(x) = \infty$. Then there exists a sequence $< x_k >$ in S of distinct points such that

$$v(x_k) > k\, u(x_k) \text{ for each } k \geq 1. \qquad (6)$$

Now, taking $e \in E$ with $\| e \| = 1$.

We define $\phi : S \to E$ by

$$\phi(x) = \begin{cases} (\xi(x))^{-1} k^{-1/v(x)} e, \text{ for } x = x_k, k \geq 1 \\ \theta, \text{ otherwise.} \end{cases} \qquad (7)$$

Then for each $x = x_k$, $k \geq 1$, we have

$$\| \xi(x)\ \phi(x) \|^{v(x)} = \| k^{-1/v(x_k)} e \|^{v(x_k)}$$
$$= \frac{1}{k} \| e \|^{v(x_k)}$$
$$\leq \frac{1}{k} A [\| e \|^{I(v)}]$$

and

$\| \xi(x)\ \phi(x) \|^{v(x)} = 0$, for $x \neq x_k$, $k \geq 1$.

This shows that $\phi \in c_0(S, (E, \|\cdot\|), \xi, v)$.

But on the other hand for each $k \geq 1$ and in view of (6) and (7), we have

$$\| \xi(x_k)\ \phi(x_k) \|^{u(x_k)} = \| k^{-1/v(x_k)} e \|^{u(x_k)}$$
$$= k^{-u(x_k)/v(x_k)} \| e \|^{u(x_k)}$$
$$> k^{-1/k} \geq e^{-1/2} .$$

This shows that $\phi \notin c_0(S, (E, \|\cdot\|), \xi, u)$, a contradiction.

For the sufficiency of the condition, assume that $\limsup_x z(x) < \infty$. Then there exists a constant $d > 0$ such that $v(x) < d\, u(x)$ for all but finitely many $x \in S$.

Let $\phi \in c_0(S, (E, \|\cdot\|), \xi, v)$ and $\varepsilon > 0$. Then for $0 < \eta < 1$ with $\eta^{1/d} < \varepsilon$, there exists $J \in \mathcal{F}(S)$ satisfying

$$\| \xi(x)\ \phi(x) \|^{v(x)} < \eta < 1, \text{ for each } x \in S - J.$$

and so

$$\| \xi(x)\ \phi(x) \|^{u(x)} \leq [\| \xi(x)\ \phi(x) \|^{v(x)}]^{1/d}$$
$$< \eta^{1/d} < \varepsilon, \text{ for each } x \in S - J.$$

This clearly implies that $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$ and hence $c_0(S, (E, \|\cdot\|), \xi, v) \subset c_0(S, (E, \|\cdot\|), \xi, u)$.

The proof is now complete.

On combining the Theorems 3 and 4, one obtain

Theorem 5. If $u, v \in \ell_\infty(S, \mathbf{R}^+)$ and $\xi \in s(S, \mathbf{C} - \{0\})$, then

$c_0(S, (E, \|\cdot\|), \xi, u) = c_0(S, (E, \|\cdot\|), \xi, v)$

if and only if $0 < \liminf_x z(x) \leq \limsup_x z(x) < \infty$.

Corollary 6. Assume that $u \in \ell_\infty(S, \mathbf{R}^+)$ and $\xi \in s(S, \mathbf{C} - \{0\})$, then

(i) $c_0(S, (E, \|\cdot\|), \xi) \subset c_0(S, (E, \|\cdot\|), \xi, u)$ if and only if $\liminf_x u(x) > 0$;

(ii) $c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \xi)$ if and only if $\limsup_x u(x) < \infty$; and

(iii) $c_0(S, (E, \|\cdot\|), \xi, u) = c_0(S, (E, \|\cdot\|), \xi)$ if and only if $0 < \liminf_x u(x) \leq \limsup_x u(x) < \infty$.

Proof: The proof follows easily if we take $u(x) = 1$ for all x and v is replaced by u in Theorems 3, 4 and 5 respectively.

Theorem 7: If $u \in \ell_\infty(S, \boldsymbol{R}^+)$ and $\xi, \mu \in s(S, C-\{0\})$, then
$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \mu, u)$
if and only if $< w(x) >_{x \in S}$ has positive limit inferior.

Proof: For the necessity of the condition, assume that
$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \mu, u)$
but $\liminf_x w(x) = 0$. Then we can find a sequence $< x_k >$ of distinct points in X such that for every $k \geq 1$,

$$k|\xi(x_k)|^{u(x_k)} < |\mu(x_k)|^{u(x_k)}. \qquad (8)$$

We now choose $e \in E$ such that $\| e \| = 1$ and define $\phi : S \to E$ by

$$\phi(x) = \begin{cases} (\xi(x))^{-1} k^{-1/u(x)} e, & \text{for } x = x_k, k = 1, 2, 3, \ldots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \qquad (9)$$

Then for each $k \geq 1$, we have

$$\| \xi(x_k)\, \phi(x_k) \|^{u(x_k)} = \| k^{-1/p(x_k)} e \|^{u(x_k)}$$
$$= \frac{1}{k} \| e \|^{u(x_k)}$$
$$\leq \frac{1}{k} A [\| e \|^{L(u)}] \to 0, \text{ as } k \to \infty$$

and

$$\| \xi(x)\phi(x)\|^{u(x)} = 0, \text{for } x \neq x_k, k \geq 1.$$

Thus for a given $\varepsilon > 0$, we can find a finite subset J of S satisfying

$$\| \xi(x)\, \phi(x) \|^{u(x)} < \varepsilon, \text{ for all } x \in S - J.$$

This clearly shows that $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$. But for each $k \geq 1$, in view of (8) and (9) we have

$$\| \mu(x_k)\, \phi(x_k) \|^{u(x_k)} = \| \mu(x_k) (\xi(x_k))^{-1} k^{-1/u(x_k)} e \|^{u(x_k)}$$
$$= \left(\frac{1}{k} \left| \frac{\mu(x_k)}{\xi(x_k)} \right|^{u(x_k)} \right) \| e \|^{u(x_k)}$$
$$> 1,$$

which is independent of k. This shows that $\phi \notin c_0(S, (E, \|\cdot\|), \mu, u)$, a contradiction.

For the sufficiency of the condition, assume that $\liminf_x w(x) > 0$. Then there exists a constant $m > 0$ such that $m |\mu(x)|^{u(x)} < |\xi(x)|^{u(x)}$ for all but finitely many $x \in S$.

Let $\phi \in c_0(S, (E, \|\cdot\|), \xi, u)$, and $\varepsilon > 0$.
Then there exists $J \in \mathcal{F}(S)$ such that

$$\| \xi(x)\, \phi(x) \|^{u(x)} < \varepsilon, \text{ for each } x \in S - J.$$

Then, we have

$$\| \mu(x)\, \phi(x) \|^{u(x)} = [|\mu(x)|\, \| \phi(x) \|]^{u(x)}$$
$$\leq \frac{1}{m} |\xi(x)|\, \| \phi(x) \|^{u(x)}$$
$$< \frac{\varepsilon}{m}, \text{ for each } x \in S - J.$$

Therefore we easily get $\phi \in c_0(S, (E, \|\cdot\|), \mu, u)$ and proves that
$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \mu, u)$.
This completes the proof.

Theorem 8: If $\xi, \mu \in s(S, \boldsymbol{C} - \{0\})$, $u \in \ell_\infty(S, \boldsymbol{R}^+)$ and $v : S \to \boldsymbol{R}^+$, then
$c_0(S, (E, \|\cdot\|), \xi, u) \subset c_0(S, (E, \|\cdot\|), \mu, v)$
if and only if
(i) $\liminf_x z(x) > 0$; and (ii) $\liminf_x w(x) > 0$.

Proof: Proof of the theorem follows immediately from the Theorems 3 and 7.

In the following example, we show that $c_0(S, (E, \|\cdot\|), \xi, u)$ may strictly be contained in $c_0(S, (E, \|\cdot\|), \mu, v)$ inspite of the satisfaction of (i) and (ii) of Theorem 8.

Example 9.
Let S be any set and $< x_k >$ be a sequence of distinct points of S. Consider $e \in E$ with $\| e \| = 1$ and define $\phi : S \to E$ by

$$\phi(x) = \begin{cases} \frac{1}{k^k} e, & \text{if } x = x_k, k = 1, 2, 3, \ldots, \text{and} \\ \theta, & \text{otherwise.} \end{cases} \qquad (10)$$

Further, if $x = x_k$, we define $u(x_k) = \frac{1}{k}$, if k is odd integer, $u(x_k) = \frac{1}{k^2}$, if k is even integer, $v(x_k) = \frac{1}{k}$ for all values of k,
$\xi(x_k) = 3^k$, $\mu(x_k) = 2^k$ for all values of k; and
$u(x) = \frac{1}{2}$, $v(x) = 1$, $\xi(x) = 3$, $\mu(x) = 2$ therwise.

Then for $x = x_k$ and $k \geq 1$, we have

$$w(x_k) = \left| \frac{\xi(x_k)}{\mu(x_k)} \right|^{u(x_k)} = \left| \frac{3^k}{2^k} \right|^{k^{-1}} = \frac{3}{2}, \text{ if } k \text{ is odd integer;}$$

$$w(x_k) = \left| \frac{\xi(x_k)}{\mu(x_k)} \right|^{u(x_k)} = \left| \frac{3^k}{2^k} \right|^{k^{-2}} = \left(\frac{3}{2} \right)^{1/k}, \text{ if } k \text{ is}$$

even integer and

$$w(x) = \left| \frac{\xi(x)}{\mu(x)} \right|^{u(x)} = \left(\frac{3}{2} \right)^{1/2} \text{otherwise.}$$

Thus, $\lim\inf_x w(x) > 0$, i.e.; condition (ii) of Theorem 11 is satisfied. Further,

$z(x_k) = \dfrac{v(x_k)}{u(x_k)} = 1$, if k is odd integer,

$z(x_k) = \dfrac{v(x_k)}{u(x_k)} = k$, if k is even integer and

$z(x) = \dfrac{v(x)}{u(x)} = 2$ otherwise.

Therefore $\lim\inf_x z(x) = 1 > 0$, i.e.,the condition (i) of Theorem 8 is also satisfied.

Then for any $k \geq 1$, we have

$$\| \mu(x_k)\ \phi(x_k) \|^{v(x_k)} = \| 2^k k^{-k} u \|^{1/k} = \frac{2}{k} \| u \|^{1/k} \leq \frac{2}{k} A [\| u \|]$$

and

$\| \mu(x)\ \phi(x) \|^{v(x)} = 0$, if $x \neq x_k$, for $k \geq 1$, which shows that $\phi \in c_0(S, (E, \|.\|), \mu, v)$.

But for even integers k and in view of (10), we get

$$\| \xi(x_k)\ \phi(x_k) \|^{u(x_k)} = \| 3^k k^{-k} u \|^{1/k} = (3/k)^{1/k} \geq \frac{1}{2}.$$

This implies that $\phi \notin c_0(S, (E, \|.\|), \xi, u)$. Thus the containment of $c_0(S, (E, \|.\|), \xi, u)$ in $c_0(S, (E, \|.\|), \mu, v)$ is strict inspite of the satisfaction of the conditions (i) and (ii) of Theorem 11.

Theorem 10. Let $u \in \ell_\infty(S, \mathbf{R}^+)$. Then for any $\xi, \mu \in s(S, \mathbf{C} - \{0\})$, $c_0(S, (E, \|.\|), \mu, u) \subset c_0(S, (E, \|.\|), \xi, u)$ if and only if $< w(x) >_{x \in S}$ has finite limit superior.

Proof: For the necessity, suppose that $c_0(S(E, \|.\|), \mu, u) \subset c_0(S, (E, \|.\|), \xi, u)$ but $\lim\sup_x w(x) = \infty$. Then there exists a sequence $< x_k >$ of distinct points in S such that for each $k \geq 1$,

$$|\xi(x_k)|^{u(x_k)} > k\, |\mu(x_k)|^{u(x_k)} \quad (11)$$

Now, we choose $e \in E$ with $\| e \| = 1$, and define $\phi: S \to E$ by $\phi(x)$

$$= \begin{cases} (\mu(x))^{-1} k^{-1/u(x)} e, \text{ for } x = x_k, k = 1, 2, 3, \dots, \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \quad (12)$$

Then for each $k \geq 1$, we have

$$\|\mu(x_k)\ \phi(x_k)\|^{u(x_k)} = \| k^{-1/u(x_k)} e \|^{u(x_k)} = \frac{1}{k} \| e \|^{u(x_k)} \leq \frac{1}{k} A [\| e \|^{I(u)}] \to 0, \text{ as } k \to \infty$$

and

$\| \mu(x)\ \phi(x) \|^{u(x)} = 0$, for $x \neq x_k$, $k \geq 1$.

Thus for a given $\varepsilon > 0$, we can find a finite subset J of S such that

$\| \mu(x)\ \phi(x) \|^{u(x)} < \varepsilon$, for all $x \in S - J$.

This shows that $\phi \in c_0(S, (E, \|.\|), \mu, u)$. But on the other hand for each $k \geq 1$ and in view of (11) and (12), we have

$$\| \xi(x_k)\ \phi(x_k) \|^{u(x_k)} = \| \xi(x_k)\ (\mu(x_k))^{-1} k^{-1/u(x_k)} e \|^{u(x_k)} = \left(\frac{1}{k} \left| \frac{\xi(x_k)}{\mu(x_k)} \right|^{u(x_k)} \| e \|^{u(x_k)} \right) > 1,$$

which is independent of k. This shows that $\phi \notin c_0(S, (E, \|.\|), \xi, u)$, a contradiction.

For the sufficiency of the condition, assume that $\lim\sup_x w(x) < \infty$. Then there exists a constant $d > 0$ such that

$|\xi(x)|^{u(x)} < d\, |\mu(x)|^{u(x)}$ for all but finitely many $x \in S$.

Let $\phi \in c_0(S, (E, \|.\|), \mu, u)$ and $\varepsilon > 0$. Then there exists $J \in \mathcal{F}(S)$ such that

$\| \mu(x)\ \phi(x) \|^{u(x)} < \varepsilon$, for each $x \in S - J$.

Then we have

$$\|\xi(x)\ \phi(x)\|^{u(x)} = [\,|\xi(x)|\, \| \phi(x) \|^{u(x)} < d\, |\mu(x)|^{u(x)} \| \phi(x) \|^{u(x)} = d \| \mu(x)\ \phi(x) \| < d\, \varepsilon, \text{ for all } x \in S - J.$$

This clearly shows that $\phi \in c_0(S, (E, \|.\|), \xi, u)$ and proves that $c_0(S, (E, \|.\|), \mu, u) \subset c_0(S, (E, \|.\|), \xi, u)$.

This completes the proof.

On combining the Theorems 7 and 10, we get the following theorem:

Theorem 11. Let $u \in \ell_\infty(S, \mathbf{R}^+)$. Then for any $\xi, \mu \in s(S, \mathbf{C} - \{0\})$, $c_0(S, (E, \|.\|), \xi, u) = c_0(S, (E, \|.\|), \mu, u)$ if and only if $0 < \lim\inf_x w(x) \leq \lim\sup_x w(x) < \infty$.

Corollary 12. For $u \in \ell_\infty(S, \mathbf{R}^+)$ and $\xi \in s(S, C - \{0\})$. Then

(i) $c_0(S, (E, \|.\|), \xi, u) \subset c_0(S, (E, \|.\|), u)$ if and only if $\lim\inf_x |\xi(x)|^{u(x)} > 0$;

(ii) $c_0(S, (E, \|.\|), u) \subset c_0(S, (E, \|.\|), \xi, u)$ if and only if $\lim\sup_x |\xi(x)|^{u(x)} < \infty$; and

(iii) $c_0(S, (E, \|.\|), \xi, u) = c_0(S, (E, \|.\|), u)$ if and only if

$$0 < \lim\inf_x |\xi(x)|^{u(x)} \leq \lim\sup_x |\xi(x)|^{u(x)} < \infty.$$

Proof: If we consider $\xi : S \to \boldsymbol{C} - \{0\}$ such that $\xi(x) = 1$ for each x, in Theorems 7 and 10 and 11 we easily obtain the assertions (i), (ii) and (iii) respectively.

In this paper,we have established some of the results that characterize paranormed structures of the class c_0 $(S, (E, ||\ .\ ||), \xi, u)$ of normed space E valued functions. In fact, these results can be used for further generalization and unification to investigate the properties of various existing normed space valued function spaces studied in Functional Analysis.

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