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Approximation of a Function Belonging to Lip ((α, r) Class by $N_{p,q}$. C_1 Summability Method of its Fourier Series

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Abstract

In this paper, an estimate for the degree of approximation of a function belonging to $Lip(\alpha, r)$ class by product summability method $N_{p,q}.C_1$ of its Fourier series has been established.

Key words: Cesàro means Lipschitz class, Nörlund means

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum is s_n. Write $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k = C_1$ means of the sequence $\{s_n\}$. If $\sigma_n \to s$ as $n \to \infty$ then we say that $\{s_n\}$ or the series $\sum_{n=0}^{\infty} u_n$ is summable to s by C_1-method. (Titchmarsh 1939)

For any two sequences $\{p_n\}$ and $\{q_n\}$ of real numbers such that $p_0 > 0, q_0 > 0$, we write

$$t_n^N = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}\, q_k\, s_k$$

where

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_k\, q_{n-k} \neq 0 \quad \forall\, n \geq 0$$

. The generalized Nörlund transform $N_{p,q}$ of the sequence $\{s_n\}$ is the sequence $\{t_n^N\}$. If $t_n^N \to s$ as $n \to \infty$, then the sequence $\{s_n\}$ is said to be summable by generalized Nörlund method $N_{p,q}$ to s (Borwein 1958). It is denoted by $t_n^N \to s\,(N_{p,q})$, as $n \to \infty$.

The $N_{p,q}$ transform of the C_1 transform defines the $N_{p,q}.C_1$ transform $\{t_n^{NC}\}$ of the partial sum $\{s_n\}$ of the series $\sum_{n=0}^{\infty} u_n$.

Thus, if

$$t_n^{NC} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k\, \sigma_k$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} \frac{p_{n-k} q_k}{k+1}\sum_{\nu=0}^{k} s_\nu \to s, \text{ as } n \to \infty$$

then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is said to be summable by $N_{p,q}.C_1$ method to s. It is denoted by $t_n^{NC} \to s\left(N_{p,q}.C_1\right)$, as $n \to \infty$.

The necessary and sufficient conditions for the $N_{p,q}$ method to be regular are

(i) $$\sum_{k=0}^{n} \left| p_{n-k}\, q_k \right| = O\left(R_n\right)$$

and

(ii) $p_{n-k} = o(R_n)$ as $n \to \infty$ for every fixed $k > 0$ for which $q_k \neq 0$.

$s_n \to s \Rightarrow C_1(s_n) = \sigma_n \to s \text{ as } n \to \infty$,

C_1 is regular

$$\Rightarrow N_{p,q}(C_1(s_n))$$
$$= N_{p,q}(\sigma_n) = t_n^{NC} \to s, \text{as } n \to \infty,$$

$N_{p,q}$ is regular

$\Rightarrow N_{p,q}.C_1$ method is regular.

It is remarkable to note that if $N_{p,q}$ method is superimposed on C_1 method then a new method of summability $N_{p,q}.C_1$ is obtained.

Some important particular cases of $N_{p,q}.C_1$ means are:

(i) $N_p.C_1$ if $q_n = 1 \;\forall n$,

(ii) $\bar{N}_q.C_1$ if $p_n = 1 \;\forall n$,

(iii) $(C,\delta)C_1$ if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$ & $q_n = 1 \;\forall n$.

Let f(x) be periodic function with period 2π, integrable in the sense of Lebesgue over $[-\pi, \pi]$ and belonging to $Lip(\alpha, r)$ class, the Fourier series of f(x) is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots (1)$$

Where

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \end{aligned}\right\}, \text{ for } n = 1, 2,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$

We define norm $\| \; \|_r$ by

$$\|f\|_r = \left\{\int_0^{2\pi} |f(x)|^r dx\right\}^{1/r}, \; r \geq 1,$$

and the degree of approximation $E_n(f)$ be given by (Zygmund 1959)

$$E_n(f; L_r^{(\alpha)}) = \min\|t_n - f\|_r,$$

where

$t_n(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is n^{th} degree trigonometric polynomial.

A function $f \in Lip\; \alpha$ if,

$$|f(x+t) - f(x)| = O(|t|^{\alpha}) \text{ for } 0 < \alpha \leq 1.$$

(Titchmarsh 1939)

A function $f \in Lip(\alpha, r)$ if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x-t)|^r dx\right\}^{1/r} \text{ (McFadden 1942)}$$
$$= O(|t|^{\alpha}), \; r \geq 1, 0 < \alpha \leq 1$$

If $r \to \infty$ in $Lip(\alpha, r)$ then it coincides with the class $Lip\; \alpha$.

We shall use the following notations:

$$\phi(x;t) = f(x+t) + f(x-t) - 2f(x)$$

$$(NC)_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{k+1}\,\frac{\sin^2(k+1)t/2}{\sin^2 t/2} \quad (2).$$

Results and Discussion

Hardy (1913) established a theorem on (C, á), (á>0) summability of the series. Harmonic summability is weaker than (C, á) summability. Iyengar (1943) proved a theorem on harmonic summability of a Fourier series. The result of Iyengar (1943) has been generalized by several researchers like Siddiqi (1948), Pati (1961), Lal and Kushwaha (2009), Rajagopal (1963), for Nörlund means. Varshney (1959), for the first time, studied the sequence $\{nB_n(x)\}$ by product summability of the form $(H, 1)C_1$. Later on

$(N, p_n)C_1$ summability of sequence $\{nB_n(x)\}$ has been studied by number of researchers like Sharma (1969 & 1970), Dwivedi (1963), Dikshit (1969). Working in the same direction, Bhattand Kathal (1966) obtained interesting results on (C,1)(E,1) summability of Fourier series and its conjugates series. These results are recently generalized by Lal & Verma (1998). Here $N_{p,q}.C_1$ summability is considered. $N_{p,q}.C_1$ summability reduces to $N_p.C_1$ if $q_n = 1\ \forall\ n$ and $\bar{N}.C_1$ if $p_n = 1\ \forall\ n$. The degree of approximation by Cesàro means, Nörlund means, $N_{p,q}$ means of a function $f \in Lip\,\alpha$, $Lip(\alpha, r)$ has been studied by number of researchers like Alexits (1928), Sahney and Goel (1973), Chandra (1975), Qureshi (1981), Qureshi and Nema (1990) and Khan (1974). But till now no work seems to have done to obtain the degree of approximation of the function $f \in Lip(\alpha, r)$ by product summability means of the form $N_{p,q}.C_1$. In an attempt to make an advance study in this direction, in this paper, an estimate for the degree of approximation of a function $f \in Lip(\alpha, r)$ class by $N_{p,q}.C_1$ means of its Fourier series has been established in the following form:

Theorem. Let $\{p_n\}$ be a non-negative, non-increasing sequence and $\{q_n\}$ be a non- negative, non-decreasing sequence such that

$$\sum_{k=0}^{n} \frac{p_{n-k}\, q_k}{k+1} = O\left(\frac{\sum_{k=0}^{n} p_{n-k}\, q_k}{n+1}\right) \quad \forall\ n \geq 0. \quad ...(3)$$

If $f : \mathbb{R} \to \mathbb{R}$ is 2π periodic, Lebesgue integrable over $[-\pi, \pi]$ and Lip (α, r) class function, then an estimate $E_n(f; L_r^{(\alpha)})$ for the degree of approximation of function f by

$$t_n^{NC} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}\, q_k\, \sigma_k \text{ i. e. } N_{p,q}.C_1$$

summability method of the Fourier series (1) is given by, for n = 0, 1, 2…,

$$E_n(f; L_r^{(\alpha)}) = \left\| t_n^{NC} - f \right\|_r = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0 < \alpha < 1 \\ O\left(\dfrac{\log(n+1)\pi}{(n+1)^{\alpha}}\right), & \alpha = 1. \end{cases}$$

Lemmas

For the proof of our theorem following Lemmas are required:

Lemma 1. Let $(NC)_n(t)$ be given by (3), then

$$(NC)_n(t) = O(n+1),$$

for $0 < t \leq 1/(n+1)$.

Proof. For $0 < t \leq 1/(n+1), \sin(k+1)t/2 \leq (k+1)t/2, \sin(t/2) \geq t/\pi,$

$$(NC)_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n} \frac{p_{n-k}\, q_k}{k+1}\, \frac{\sin^2(k+1)t/2}{\sin^2 t/2}$$

$$(NC)_n(t) \leq \frac{1}{2\pi R_n}\sum_{k=0}^{n} \frac{p_{n-k}\, q_k\, ((k+1)t/2)^2}{(k+1)\,(t/\pi)^2}$$

$$= \frac{\pi}{8R_n}\sum_{k=0}^{n} (k+1)\, p_{n-k}\, q_k$$

$$= \frac{(n+1)\,\pi}{8R_n} R_n$$

$$= O(n+1)$$

Lemma 2. Let $(NC)_n(t)$ be given by (3), then

$$(NC)_n(t) = O\left(\frac{1}{(n+1)t^2}\right), \text{ for } 1/(n+1) < t \leq \pi.$$

Proof.

For $1/(n+1) < t < \pi, \sin^2(k+1)t/2 \leq 1, \sin t/2 \geq t/\pi$ we have

$$(NC)_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n} \frac{p_{n-k}\, q_k}{k+1}\, \frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2\pi R_n}\sum_{k=0}^{n} \frac{p_{n-k}\, q_k}{k+1}\, \frac{\pi^2}{t^2}$$

$$= \frac{\pi}{2 R_n t^2} \; O\left(\frac{R_n}{n+1}\right), \quad \text{by (3)}$$

$$= O\left(\frac{1}{(n+1)t^2}\right).$$

Lemma 3. If $f \in Lip(\alpha, r)$, $1 \le r < \infty$, then

$$\left[\int_0^{2\pi} |\phi(x;t)|^r \, dx\right]^{1/r} = O\left(\left|t^{\alpha}\right|\right).$$

Proof.

$$|\phi(x;t)| = |f(x+t) + f(x-t) - 2f(x)|$$

$$\le |f(x+t) - f(x)| + |f(x-t) - f(x)|.$$

$$\text{Then,} \left[\int_0^{2\pi} |\phi(x;t)|^r dx\right]^{1/r} \le \left[\int_0^{2\pi} |f(x+t) - f(x)|^r\right]^{1/r} + \left[\int_0^{2\pi} |f(x-t) - f(x)|^r\right]^{1/r},$$

by Minkowski's inequality

$$= O\left(\left|t^{\alpha}\right|\right).$$

Lemma 4. The inequality

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_a^b g(x;t)\, dt\right|^r dx\right\}^{1/r} \le \int_a^b \left\{\frac{1}{2\pi}\int_0^{2\pi} |g(t;x)|^r dx\right\}^{1/r} dt$$

is known as generalized Minkowski's Inequality where the generalization is simply replacing a finite sum by a definite (Lebesgue) integral.

Proof of the theorem

Following Titchmarsh (1939), n^{th} partial sum $s_n(f; x)$ of the Fourier series (1) at $t = x \in [-\pi, \pi]$ is given by

$$\left(s_n(f;x) - f(x)\right) = \frac{1}{2\pi}\int_0^{\pi} \phi(x;t) \frac{\sin(n+1/2)t}{\sin t/2} \, dt.$$

The (C,1) transform i.e. σ_n of s_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n} \left(s_k(f;x) - f(x)\right) =$$

$$\frac{1}{2(n+1)\pi}\int_0^{\pi} \frac{\phi(x;t)}{\sin\frac{t}{2}} \sum_{k=0}^{n} \sin(k+1/2)t \, dt$$

$$(\sigma_n(f;x) - f(x)) =$$

$$\frac{1}{2(n+1)\pi}\int_0^{\pi} \frac{\sin^2(n+1)t/2}{\sin^2 t/2} \, \phi(x;t)\, dt.$$

Denoting $N_{p,q}$ transform of σ_n i.e. $N_{p,q}.C_1$ transform of S_n by t_n^{NC}, we have

$$\frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \{\sigma_k(f;x) - f(x)\}$$

$$= \int_0^{\pi} \frac{1}{2\pi R_n} \sum_{k=0}^{n} \frac{p_{n-k} q_k}{k+1} \frac{\sin^2(k+1)t/2}{\sin^2 t/2} \phi(x;t)\, dt$$

$$t_n^{NC}(x) - f(x) = \int_0^{\pi} (NC)_n(t)\; \phi(x;t)\, dt.$$

$$\left|t_n^{NC}(x) - f(x)\right| \le \int_0^{\pi} \left|(NC)_n(t)\right| \left|\phi(x;t)\right| dt.$$

$$\left[\int_0^{2\pi} \left|t_n^{NC}(x) - f(x)\right|^r dx\right]^{1/r}$$

$$= \left[\int_0^{2\pi}\left|\int_0^{\pi} |(NC)_n(t)| \, |\phi(x;t)| dt\right|^r dx\right]^{1/r}$$

$$\left\|t_n^{NC}(x) - f(x)\right\|_r \le \int_0^{\pi}\left\{\int_0^{2\pi} |\phi(x;t)|^r dx\right\}^{1/r} |(NC)_n(t)| dt, \text{ by}$$

generalized Minkowski's inequality and Lemma 4

$$\left\|t_n^{NC}(x) - f(x)\right\|_r = \int_0^{\pi} O(t^{\alpha})\,(NC)_n(t)\, dt, \textit{ by Lemma (3)}$$

$$= O\left(\int_0^{1/(n+1)} t^{\alpha}\,(NC)_n(t)\, dt\right) + O\left(\int_{1/(n+1)}^{\pi} t^{\alpha}\,(NC)_n(t)\, dt\right)$$

$$= O(I_1) + O(I_2), \textit{ say.}$$

Now, using Lemma 1, we have

$$I_1 =$$

$$O\left[\int_0^{1/(n+1)} t^{\alpha}\,(NC)_n(t)\, dt\right]$$

$$= O\left(\int_0^{1/(n+1)} t^{\alpha}\,(n+1)\,dt\right)$$

$$= O\left((n+1)^{-\alpha}\right). \qquad (5)$$

By Lemma 2, we have

$$I_2 = O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\alpha}}{(n+1)\,t^2}\,dt\right]$$

$$= O\left(\frac{1}{n+1}\right)\int_{1/(n+1)}^{\pi} t^{\alpha-2}\,dt$$

$$= \begin{cases} O\left(\frac{1}{n+1}\right)\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right), & 0<\alpha<1 \\ O\left(\frac{1}{n+1}\right)\left(\log\pi - \log\left(\frac{1}{n+1}\right)\right), & \alpha = 1 \end{cases}$$

$$= \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi}{n+1}\right), & \alpha = 1. \end{cases} \qquad (6)$$

Combining from (4) to (6), we have

$$\left\|t_n^{NC} - f\right\|_r = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)^{\alpha}}\right), & \alpha = 1. \end{cases}$$

This completes the proof of our theorem.

Corollaries

Following corollaries can be derived from our theorem:

Corollary 1. Let $\{p_n\}$ be a non-negative, non-increasing sequence such that

$$\sum_{k=0}^{n} \frac{p_{n-k}}{k+1} = O\left(\frac{P_n}{n+1}\right) \;\forall\; n \geq 0.$$

If $f : \mathbb{R} \to \mathbb{R}$ is 2π periodic, Lebesgue integrable over $[-\pi, \pi]$ and Lip (α, r) class function, then an estimate $E_n(f; L_r^{(\alpha)})$ for the degree of approximation of function f by

$t_n^{NC} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\,\sigma_k$ i. e. $N_p.C_1$ – summability method of the Fourier series (1) is given by, for n = 0, 1, 2 ...,

$$E_n(f; L_r^{(\alpha)}) = \left\|t_n^{NC} - f\right\|_r = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)^{\alpha}}\right), & \alpha = 1 \end{cases}$$

Corollary 2. $\{q_n\}$ be a non- negative, non-decreasing sequence such that $Q_n = \sum_{k=0}^{n} q_k \neq 0,$

$$\sum_{k=0}^{n} \frac{q_k}{k+1} = O\left(\frac{\sum_{k=0}^{n} q_k}{n+1}\right) \;\forall\; n \geq 0.$$

If $f : \mathbb{R} \to \mathbb{R}$ is 2π periodic, Lebesgue integrable over $[-\pi, \pi]$ and Lip (α, r) class function, then an estimate $E_n(f; L_r^{(\alpha)})$ for the degree of approximation of function f by

$t_n^{NC} = \frac{1}{Q_n}\sum_{k=0}^{n} q_k\,\sigma_k$ i. e. $\bar{N}_q.C_1$ – summability method of the Fourier series (1) is given by, for n = 0, 1, 2 ...,

$$E_n(f; L_r^{(\alpha)}) = \left\|t_n^{NC} - f\right\|_r = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)^{\alpha}}\right), & \alpha = 1. \end{cases}$$

Corollary 3. If $f \in Lip(\alpha, r)$ then its degree of approximation by $\left(N, \frac{1}{n+1}\right)C_1$ transform $t_n^{NC} = \frac{1}{\log(n+1)}\sum_{k=0}^{n} \frac{s_k}{n-k+1}$ of its Fourier series (1) is given by

$$E_n(f; L_r^{(\alpha)}) = \left\|t_n^{NC} - f\right\|_r = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi}{(n+1)^{\alpha}}\right), & \alpha = 1. \end{cases}$$

Example

Consider the infinite series

$$1 + 4\sum_{n=1}^{\infty} n\,(-1)^n \,. \qquad \textbf{(7)}$$

The n^{th} partial sum of (7) is given by

$$s_n = 1 + 4\sum_{k=1}^{n} k(-1)^k$$
$$= (2n+1)(-1)^n$$

and so

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}(2k+1)(-1)^k = (-1)^n .99999$$

Therefore the series (7) is not (C,1) summable. Since $\{(-1)^n\}$ is (N, p,q) summable therefore, the series (7) is $N_{p,q}.C_1$ summable. Hence the product summability $N_{p,q}.C_1$ is more powerful than the individual methods (N,p,q) and (C,1). Concequently $N_{p,q}.C_1$ means gives better approximation than individual methods (N,p,q) and (C,1).

References

Alexits, G. 1928. Über die Annäherung einer stetigen Funktion durch die Cesàroschen Mittel ihrer Fourierreihe, *Math. Ann.* **100**:264-277.

Bhatt, S. K. and P.D. Kathal. 1996. The (C, 1) (E, 1) summability of a Fourier series and its conjugate series. *Acta Cienc. Indica Math.* **22** :375–380.

Borwein, D.1958. On products of sequences. *J. London Math. Soc.* **33**:352–357.

Chandra, P. 1975. On the degree of approximation of functions belonging to the Lipschitz class. *Nanta Math.* **8**:88–91.

Dikshit, H. P. 1969. Summability of a sequence of Fourier coefficients by a triangular matrix transformation. *Proc. Amer. Math. Soc.* **21**:10–20.

Dwivedi, G. K. 1971. On a sequence of Fourier coefficients, Ann.Soc. Math. Polonae, Series I, *Commentationes Mathematicae* **15**: 61-66.

Hardy, G. H. 1913. On the summability of Fourier series, *Proc. London Math. Soc.* **12**:365-372.

Lyengar, K. S. K. 1943. A Tauberian theorem and its application to convergence of Fourier series. *Proc. Indian Acad. Sci., Sect. A.* **18:**81–87.

Khan, H. H. 1974. On the degree of approximation of functions belonging to class Lip (á,p). *Indian J. Pure Appl. Math. Indian J. Pure Appl. Math.* **5**:132–136.

Lal, Shyam and J.K. Kushwaha. 2009. Degree of approximation of Lipschitz function by product summability method. *Int. Math. Forum* **4** (44):2101–2107.

Lal, Shyam and S. Verma. 1998. The (C,1)(E,1) summability of a Fourier series and its *Vijnana Parishad Anusandhan Patrika* **41** :265–276. (In India).

McFadden, L. 1942. Absolute Nörlund summability**,** *Duke Math. J.* **9:** 168–207.

Pati, T. 1961. A generalization of a theorem of Iyengar on the harmonic summability of Fourier series. *Indian J. Math.* **3**:85–90.

Qureshi, K. 1981. On the degree of approximation of a periodic function ff by almost Nörlund means. *Tamkang J. Math.* **12**: 35–38.

Qureshi, K. and H.K Nema. 1990. A class of functions and their degree of approximation. *Ganita* **41**: 37–42.

Rajagopal, C. T. 1963. On the Nörlund summability of Fourier series. *Proc. Cambridge Philos. Soc.* **59:**47–53.

Sahney, B. N. and D.S. Goel. 1973. On the degree of approximation of continuous functions. *Ranchi Univ. Math. J.* **4**:50–53.

Sharma, R. M. 1969. On a sequence of Fourier coefficients. *Bull. Calcutta Math. Soc.* **61**:89–93.

Sharma, R. M. 1970. On (N, pn) C1 summability of the sequence {nBn (x)}. *Rend. Circ. Mat. Palermo (2)* **19**: 217–224.

Siddiqi, J. A. 1948. On the harmonic summability of Fourier series. *Proc. Indian Acad. Sci., Sect. A.* **28**:527–531.

Titchmarsh, E. C. 1939. *The Theory of functions, Second Edition*, Oxford University Press.

Varshney, O. P. 1959. On a sequence of Fourier coefficients, *Proc. Amer. Math. Soc.* **10:**790 - 795.

Zygmund, A. 1959. *Trigonometric series*, Cambridge University Press.